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SOME PROBLEMS IN THE THEORY OF NUMBERS.

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REGULAR, POSITIVE DEFINITE, TERNARY QUADRATIC FORMS.

1. I consider in this paper positive definite quadratic forms, in three integral variables, of the type

$$f = ax^2 + by^2 + cz^2 + ryz + szx + txy,$$

where a, b, c, r, s, t are integers without a common factor. Such a form will, for brevity, be called simply a form.

The form f is said to be regular if it represents every positive integer n for which the congruence $f \equiv n \pmod{m}$ is soluble for every positive integer m. Regularity is thus essentially a property of a class of forms, equivalent under integral unimodular transformations. The simplest instance of it is the classical three square theorem: the form $x^2 + y^2 + z^2$ does not represent any number of the form $4^l(8h + 7)$, since the congruence

$$x^2 + y^2 + z^2 \equiv 7.4^l \pmod{8.4^l}$$

is insoluble; but it does represent all other positive integers. Regular forms have been investigated by Dickson* and by Jones and Pall+; these writers confined themselves almost entirely to diagonal forms, that is, forms with $r = s = t = 0$.

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*Annals of Math. 28 (1927), 333–341.

+Acta Math. 70 (1939), 165–191.

2. I write

$$(1)\quad d = d(f) = \tfrac{1}{2}\begin{vmatrix} 2a & t & s \\ t & 2b & r \\ s & r & 2c \end{vmatrix} = 4abc + rst - ar^2 - bs^2 - ct^2,$$

and denote by $\omega = \omega(f)$ the highest common factor of the first minors of the determinant 2d. It is clear that

$$(2)\qquad 4d \equiv 0 \pmod{\omega^2}.$$

In case r, s, t are all even, the classical invariants D, Ω are d/4, $\omega/4$.

I prove the following theorems:

THEOREM 1. (a) If d(f) is not square-free, there exists a form $h = h(f)$ such that:

(i) $d_0(f) = d(h)$ is square-free, and a divisor of d(f), and

(ii) the regularity of h is a necessary condition for that of f.

(b) If d(f) is divisible by the squares of two or more primes, one of which is p, then there exists a form $h_p = h_p(f)$ such that:

(i) $d(h_p) \not\equiv 0 \pmod{q^2}$ for any prime $q \neq p$,

(ii) $d(h_p)$ is a divisor of d(f),

(iii) $d(h_p)$, and $\omega(h_p)$ are divisible by precisely the same powers of p as d(f) and $\omega(f)$ respectively, and

(iv) the regularity of h_p is a necessary condition for that of f.

THEOREM 2. Every regular form h, with square-free d, is equivalent to one or other of the twenty-four forms $h_1, \ldots, h_{24}$ listed in Table I (Appendix) (for each of which d is at most 78 and does not divide by 19, nor by any prime exceeding 23).

THEOREM 3. Suppose that f is regular and that the prime p does not divide $\omega(f)$, but p^ν divides $d(f)$. Then $\nu = 0$ if $p = 19$ or if $p \geqslant 29$, $\nu \leqslant 1$ if $p \geqslant 7$, and $\nu \leqslant 6, 4, 2$ for $p = 2, 3, 5$.

THEOREM 4. Suppose that f is regular and that $\omega(f)$ is divisible by an odd prime p, but not by p^2, and that p^ν divides $d(f)$. Then $p \leqslant 23$, $p \neq 19$, and

$$\nu \leqslant \begin{cases} 2 & \text{if } p = 11, 13, 17 \text{ or } 23, \\ 3 & \text{if } p = 5 \text{ or } 7, \\ 5 & \text{if } p = 3. \end{cases}$$

THEOREM 5. If f is regular, then $\omega(f)$ is not divisible by 27, nor by the square of any prime greater than 3; and $d(f)$ is not divisible by 3^6.

THEOREM 6. If f is regular and classic (that is, r, s, t are all even), then $\Omega(f)$ cannot be divisible by 2^6, nor $D(f)$ by 2^{12}.

Theorems 2 to 6 show that, if f is regular and p^ν ($\nu \geqslant 1$) divides $d(f)$, then p and ν are both bounded. It easily follows that there is only a finite number of classes of regular forms. A complete enumeration would be very laborious, but Tables I to III in the Appendix show all possible ones with d not divisible by 4; including two whose regularity I have not been able to prove.

3. The letters f, g, h denote forms. All other letters denote rational integers, n being positive, while p, q, P, Q, with or without accents and suffixes, are primes. For odd p, $\left(\frac{x}{p}\right)$ is the Legendre symbol, and R_p, N_p run through all integers n from 1 to $p-1$ for which $\left(\frac{n}{p}\right) = 1, -1$ respectively. In case p divides ω, it is well known that $\left(\frac{n}{p}\right)$ has the same value, denoted as usual by $\left(\frac{f}{p}\right)$, for all n representable by f and not divisible by p.

The solubility of the congruence $f \equiv n \pmod{m}$ need only be considered for $m = p^k$, p being a factor of d, and $k = 1, 2, 3 \ldots$ Those n for which the congruence

$$(3) \qquad f \equiv n \pmod{p^k}$$

is, for a given p and some k, insoluble will be said to constitute the p-progressions of f. The p-progressions, collectively called the progressions of f, are, as is well known, an infinite system of arithmetical progressions.

I denote by $C = 4ab - t^2$ the negative of the discriminant of the binary form $f(x, y, 0)$. Clearly

(4) $C > 0$, $C \equiv 0$ or 3 (mod 4), $C \equiv 0$ (mod ω).

4. The general plan of the work is as follows:

(i) We prove Theorem 1, making systematic use of a method, by which the regularity of a form with non square-free d can be shown to imply that of a simpler form. The principle of this method is well known*.

(ii) Theorem 1 reveals the fundamental importance of forms with square-free d, which accordingly we next investigate (Theorem 2). The task is made easier by the simplicity, and low density, of the p-progressions of these forms, especially for $p = 2$.

(iii) We now come back to the forms with non square-free d, and prove Theorems 3 to 6. In so doing we may assume that no prime $q \neq p$ is a repeated factor of d; for if it were, we could by Theorem 1(b) work with $h_p(f)$ instead of f. The work is further simplified by what we know from Theorem 2 about the q-progressions of f. In fact, if p is the repeated prime factor of d, and $q \neq p$ is a simple factor of d,

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* See, e.g., Burton W. Jones, An extension of Meyer's theorem on indefinite ternary quadratic forms, <u>Canadian Journal of Mathematics</u>, 4 (1952), 120-8 (definition of p-related form, p. 120)

then we know $q \leqslant 23, \neq 19$. Further, Table I shows that there are no 23-progressions, unless $p = 23$, while if $p \neq 11, 17$ there are either no 11-, 17-progressions or no others (except p-).

The work is thus considerably simplified. It has however to be done in essentially three different cases, corresponding to the three ways by which a repeated factor of d can be removed.

(iv) The detailed method of investigating regular forms f with invariants d, ω satisfying given conditions depends on considering the successive minima of f. We first seek a number n for which we know that (3) can never be insoluble, and which is square-free. We may then suppose $a \leqslant n$; for the form may be so transformed as to have its minimum as leading coefficient. It may be necessary to use alternative values of n. Now we seek a number n' satisfying the same conditions as n, but not equal to n; we may suppose $n' > n$, and so, being square-free, n' is not expressible as ax^2, so that it gives a bound for the second minimum of f (which may be taken as b). Lastly, a number n'', satisfying the same conditions as n, n', and further, not representable by $f(x, y, 0)$, gives a bound for the third minimum of f, and so for d.

5. <u>The congruence properties and progressions of f.</u>

It is easily shown that, for an odd prime p and arbitrary positive k, we may transform f so that $r \equiv s \equiv t \equiv 0 \pmod{p^k}$, that is, we may have

(5) $\quad f \equiv ax^2 + by^2 + cz^2 \pmod{p^k}$.

If $p = 2$, (5) may still be possible; if not, we may have either

(6) $\quad f \equiv ax^2 + by^2 + byz + cz^2 \pmod{2^k}$,

with b even, if f is classic, or

(7) $\quad f \equiv ax^2 + txy + by^2 + cz^2 \pmod{2^k}$,

if f is non-classic. In case (7) we note that if a and b are both odd, that is, if $C \equiv 3 \pmod 8$, $f(x, y, 0)$ will not represent $2 \pmod 4$ nor $8 \pmod{16}$; whence, if further $c \equiv 2 \pmod 4$, f will not represent $c + 8 \pmod{16}$. In this case we have $d \equiv 8 - c \pmod{16}$; thus f fails to represent numbers $\equiv -d \pmod{16}$.

From the foregoing results* we may readily deduce the following:

LEMMA 1. (a) The p-progressions of f

(i) do not exist if p does not divide d, nor if $\left(\frac{-C}{p}\right) = +1$ (for odd p), or $C \equiv 7 \pmod 8$ (for $p = 2$),

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*These results, and the Lemma, are contained in substance in Minkowski's Paris prize essay: Mémoires présentés à l'Académie des Sciences, 29 (1884), No. 2, or Ges. Abh., I, 3 - 144.

(ii) include no integers not divisible by p, unless p divides ω.

(b) If p divides ω, then the numbers not divisible by p that belong to the p-progressions are those congruent to

(i) either R_p or N_p, but not both, if p is odd,

(ii) three at most of 1, 3, 5, 7 (mod 8), if $p = 2$.

(c) If p divides d and $\left(\frac{-C}{p}\right) = -1$ (for odd p), or $C \equiv 3 \pmod 8$ (for $p = 2$), then p-progressions exist; if further p^2 does not divide d, then they consist of the numbers $\equiv -p^{2k}R_p d \pmod{p^{2k+2}}$ or $-2^{2k}d \pmod{2^{2k+4}}$, for all $k \geq 0$, and no others.

6. We shall now deduce Theorem 1 from the following

LEMMA 2. Suppose that $d(f) \equiv 0 \pmod{p^2}$. Then there exists a form $g = g_p(f)$ such that:

(i) $d(f)/d(g) = p, p^2$ or p^4,

(ii) $\omega(f)/\omega(g) = 1, p$ or p^2, and

(iii) the regularity of g is a necessary condition for that of f.

Proof. We first transform f so that (5), (6) or (7) holds, with a sufficiently large* k, and then consider several cases.

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* g does not depend on k, and is indeed unique up to equivalence. (We need not take k greater than 2.) But we do not need this fact, and the properties (i) and (ii) are clearer if we suppose k so large that p^k does not divide $d(f)$.

Case 1, p odd, p^2 divides ω. We may suppose in (5) that $b \equiv c \equiv 0 \pmod{p^2}$, $a \not\equiv 0 \pmod{p}$; then we define

$$(8) \qquad g(x, y, z) = p^{-2}f(px, y, z),$$

giving

$$(9) \qquad g \equiv ax^2 + bp^{-2}y^2 + cp^{-2}z^2 \pmod{p^{k-2}}.$$

It is clear from (8) and (9) that $d(g) = p^{-4}d(f)$, $\omega(g) = p^{-2}\omega(f)$. To prove (iii), suppose f is regular. If $g \equiv n \pmod{m}$ is soluble for every m, then so too is $f \equiv p^2n \pmod{m}$, as we see from (8). By the regularity of f, $f = p^2n$ must be soluble. But with our hypotheses (5) shows that $f = p^2n$ can only be soluble with $x \equiv 0 \pmod{p}$. Thus by (8) $g = n$ must be soluble. Hence if f is regular g must be, and (iii) holds.

Case 2, p an odd, simple factor of ω. We may now suppose in (5) that $b \equiv c \equiv 0 \pmod{p}$, $b \not\equiv 0 \pmod{p^2}$, $a \not\equiv 0 \pmod{p}$. We define

$$(10) \qquad g(x, y, z) = p^{-1}f(px, y, z)$$

$$\equiv pax^2 + bp^{-1}y^2 + cp^{-1}z^2 \pmod{p^{k-1}}.$$

(i) and (ii) are again clear; the first or second case of (ii) arises according as p^2 does or does not divide c, that is, according as p^3 does or does not divide d. The proof of (iii) is similar to that in case 1.

Case 3, p odd and not a factor of ω. We assume in (5) $ab \not\equiv 0 \pmod{p}$, $c \equiv 0 \pmod{p}$. We define

$$(11)\quad g(x, y, z) = p^{-2}f(px, py, z)$$

$$\equiv ax^2 + by^2 + cp^{-2}z^2 \pmod{p^{k-2}}.$$

The results (i) and (ii) are again clear. If we assume ab to be a non-residue of p, then f can represent a number p^2n, if at all, only with $x \equiv y \equiv 0 \pmod p$, and so we prove (iii) as in case 1.

But if $\left(\frac{ab}{p}\right) = +1$, the argument to prove (iii) is different: we see from (11) that g represents every integer represented by f, hence g must be regular with f if it has the same progressions as f. Now by Lemma 1(a)(i), neither f nor g has any p-progressions, while by (11) it is clear that they have the same q-progressions for any $q \neq p$.

<u>Case 4, $p = 2$, f classic.</u> We may suppose a odd, and we write $f(2x + by + cz, y, z) = 4g$ or $2g$, according as the coefficients of this expression, which are clearly all even, are or are not all divisible by 4. The proof is as in case 1 or 2.

<u>Case 5, $p = 2$, f non-classic.</u> We have $c \equiv 0 \pmod 4$ in (7) and we define

$$(12)\quad g(x, y, z) = \tfrac{1}{4}f(2x, 2y, z)$$

$$\equiv ax^2 + txy + by^2 + \tfrac{1}{4}cz^2 \pmod{2^{k-2}}.$$

Distinguishing the cases $C \equiv 3, 7 \pmod 8$, we argue as in case 3.

Theorem 1 follows on applying Lemma 2 repeatedly to all, or all but one, of the repeated prime factors of d(f).

7. Reduction of forms.

LEMMA 3. Every form f is equivalent to one satisfying

$$(13)\qquad f \geqslant \begin{cases} a, \text{ if } & x, y, z \neq 0, 0, 0, \\ b, \text{ if } & y, z \neq 0, 0, \\ c, \text{ if } & z \neq 0; \end{cases}$$

$$(14)\qquad 0 \leqslant t \leqslant a, \quad |s| \leqslant a, \quad |r| \leqslant b;$$

$$(15)\qquad d \leqslant Ce \leqslant 4abc.$$

If $(13)_1$, $(13)_2$ and $(14)_1$ hold, then

$$(16)\qquad f \geqslant d/C, \quad \text{for} \quad z \neq 0.$$

Proof. The inequalities (12) express that the form is reduced so as to have a, b, c as its successive minima; this is well known to be possible*. The inequalities (14) are obtainable by putting $x, y, z = 1, \pm 1, 0$; $1, 0, \pm 1$; $0, 1, \pm 1$. The possibility $t < 0$ can be avoided by the transformation $x, y, z = x', -y', z'$. Then (15) follows from

*It is not easy to find a reference, but the result is easily obtained by the method of Korkine and Zolotareff, Math. Annalen, 6 (1872), 336.

$$Cc - d = ar^2 - trs + bs^2 = f(r, -s, 0) \geqslant 0.$$

Now (16) follows from $(13)_3$ and $(15)_1$; but it clearly remains true for any equivalent form with the same a, b, t. We shall work mainly with $(13)_1$, $(13)_2$ and (16); we often find a more convenient form of the class by dropping $(13)_3$.

8. Reduced regular forms with $\omega = 1$. The lemmas of this section are required for the proof of Theorem 2, but as we shall not need the full force of the hypothesis of that Theorem that d(f) is square-free, they will also be used in Theorem 3.

LEMMA 4. Suppose f is regular and satisfies $(13)_1$, (13 and $(14)_1$, (f) = 1, and neither 4 nor 9 divides d(f). Then a = 1; and if t = 0, b = 1 or 2, while if t = 1, b = 1, 2, 3, or 5.

Proof. By Lemma 1(a)(ii) and the hypothesis $\omega = 1$, f must represent 1; hence by (13) we have a = 1. Similarly we see that f represents 2, and deduce $b \leqslant 2$, unless f has 2-progressions, and $d \equiv 14 \pmod{16}$ (Lemma 1(a) and (c)).

Assuming that $d \equiv 14 \pmod{16}$, the 2-progressions of f do not include the number 6, while by Lemma 1(c), the 3-progressions of f do not include both 3 and 6, and neither 3 nor 6 can by Lemma 1(a)(ii) belong to a p-progression for $p > 3$. Thus f must if regular represent either 3 or 6, and so by $(13)_2$ $b \leqslant 6$.

If t [illegible] $= 1$, it remains only to show that we cannot have $b = 4$ or 6; but if so, $C \equiv 7 \pmod 8$, so there are no 2-progressions by Lemma 1(a)(i), whereas by (13) $f = 2$ is insoluble, and so f is irregular.

If $t = 0$, we have still to exclude the four possibilities $b = 3, 4, 5, 6$. The first case $b = 3$ is inconsistent with $d \equiv 14 \pmod{16} \equiv 2 \pmod 4$. If $b = 4$, f represents 8 but not 2; but by Lemma 1(c) 8 would, with 2, belong to the 2-progressions. If $b = 5, 6$, f does not, by (13), represent 3, 5, although by Lemma 1(a)(i) there are no 3-, 5-progressions; so f would be irregular. Thus the proof is complete.

LEMMA 5. In the six cases of Lemma 4, assuming the hypotheses of that Lemma and also that $d(f)$ has at most one repeated prime factor (> 3), we have the following upper bounds for $d(f)$:

$C = 3,$	$4,$	$7,$	$8,$	$11,$	$19,$
$d \leqslant 66,$	$60,$	$42,$	$168,$	$66,$	$114.$

Proof. The last case is easiest; the proof of Lemma 4 shows that f must represent 2, 3, or 6, but $f(x, y, 0) = x^2 + xy + 5y^2$ represents none of these integers. So one of them (necessarily 6) must be represented with $z \neq 0$, whence by [illegible] (16) $d \leqslant 6C = 114$.

Again in the fifth case f must represent 6, which $f(x, y, 0) = x^2 + xy + 3y^2$ cannot, so $d \leqslant 6C = 66$. For there are no 3-progressions, and so a modification of the argument of Lemma 4 shows that f represents 2 or 6, clearly not 2. In case $C = 7$, $f(x, y, 0) = x^2 + xy + 2y^2$, ~~but~~ and a similar argument to the foregoing shows that though $f(x, y, 0)$ represents neither 3 nor 6, f must represent one of these integers, hence again $d \leqslant 6C$.

In case $C = 4$, $f(x, y, 0) = x^2 + y^2$, f has no 5-progressions, hence it represents either 3 or 15, and $d \leqslant 15C = 60$.

If $C = 8$, suppose first that d does not divide by 7^2. Then since $\left(\frac{3}{7}\right) = -1$, and there are no 3-progressions, Lemma 1(c) shows that f represents either 7 or 21, but $f(x, y, 0) = x^2 + 2y^2$ does not, so $d \leqslant 21C = 168$. If however d divides by 7^2, then by hypothesis d does not divide by 5^2, and by considering the numbers 5, 15 we obtain a better result.

Lastly, if $C = 3$, $f(x, y, 0) = x^2 + xy + y^2$, we have $d \leqslant 6$ unless f fails to represent 2. If so, neither 10 nor 22 can belong to a 2-progression, and we obtain the result by considering the numbers 11, 22 ~~if~~ unless d divides by 11^2, in which case we use 5, 10 instead. Thus the proof is complete.

LEMMA 6. Assume the hypotheses of Lemma 5, and further that in case of ambiguity the least possible value of C is considered. Then the only possibilities for C, $f(x, y, 0)$ and d are:

C	$f(x, y, 0)$	d
3	$x^2 + xy + y^2$	2, 3, 5, 6, 14, 30.
4	$x^2 + y^2$	6, 7, 10, 11, 15, 42.
7	$x^2 + xy + 2y^2$	10, 13, 17, 21, 33,
8	$x^2 + 2y^2$	15, 21, 22, 50, 70.
11	$x^2 + xy + 3y^2$	30, 46.
19	$x^2 + xy + 5y^2$	78.

Proof. In case $C = 3$, we note first that $d = 3c - r^2 + rs - s^2 \equiv 0$ or $2 \pmod 3$. Next, if f represents 2, we have $d \leq 6$, while if not, $d \equiv 14 \pmod{16}$, as we have seen. With the bound $d \leq 66$ of Lemma 5, it remains only to show that $d \neq 46, 62$. For either of these d however, we should have no 5-progressions, whereas f would not represent 5 except, by (16), with $z = 0$, which is clearly impossible.

In case $C = 4$, $d = 4c - r^2 - s^2 \equiv 2$ or $3 \pmod 4$. We exclude the cases $d = 2, 3$, because there are equivalent forms with $C = 3$. If $p \equiv -1 \pmod 4$, $x^2 + y^2$ does not represent p, but f must do so, giving by (16) $d \leq 4p$, unless $d \equiv 0 \pmod p$. These arguments, with $d \leq 60$, exclude all cases other than those stated, except $d = 14$. If however $d = 14$, $f = x^2 + y^2 + xz + yz + 4z^2$ is irregular; it does not represent 7.

Similar arguments in the other four cases exclude all unwanted values of d, except the following, for which we exhibit in each case an integer n, in no progression of f, which is not representable by f:

C	d	f		n
7	14	$x^2 + xy + 2y^2 +$	$2z^2$	5
	19		$xz + 3z^2$	10
8	23	$x^2 + 2y^2 +$	$yz + 3z^2$	23
	26		$xz + 2yz + 4z^2$	5
	29		$xz + yz + 4z^2$	87
	30		$xz + 4z^2$	7
	31		$yz + 4z^2$	31
	34		$xz + 2yz + 5z^2$	10
	37		$xz + yz + 5z^2$	185
	38		$xz + 5z^2$	14
	39		$yz + 5z^2$	91
	55		$yz + 7z^2$	15
11	62	$x^2 + xy + 3y^2 + 2yz + 6z^2$		26

[It simplifies the calculation to note that in case C = 8 we have $8f = (4y + rz)^2 + 2(2x + sz)^2 + dz^2$.]

9. <u>Proofs of Theorems 2, 3.</u> Excluding the case $d = 50$, each of the twenty-four other cases of Lemma 6 gives just one class of forms with square-free d, which are those shown in Table I. Thus Theorem 2 is proved. We shall see later that these twenty-four forms are all regular.

To prove Theorem 3, we may, by Theorem 1(b), assume

(17) $d(f) \not\equiv 0 \pmod{q^2}$, if $q \neq p$.

Then if $p \geqslant 5$ the hypotheses of Lemma 6 are satisfied, and the conclusion of the Theorem may be verified from the table in that Lemma.

If however $p = 2$ or 3, we have to rework Lemmas 4 to 6 on the new hypothesis (17). To do this in the first place would have been difficult, but now it is made easier by the results of Theorem 2. Note for example that, by (11) and (12), $h(f)$, which must be one of the forms in Table I, has precisely the same q-progressions as f, for any $q \neq p$. Now every form in Table I represents either 2 or 7, so on reworking Lemma 4 for a form with $d \equiv 0 \pmod 9$ and satisfying (17), we have $a = 1$ and $b \leqslant 7$.

I omit the necessary calculations, here and in much of what follows, as they are long and the principle has been made clear enough.

10. <u>Proof of Theorem 4.</u> Under the hypotheses of this theorem, the prime p is dealt with solely by case 2 of Lemma 2. Hence after repeated applications of that Lemma, p remains a simple factor of $d_0(f)$, that is, of one of the d in Table I, so p cannot be 19, nor > 23.

The bounds for the exponent ν can be obtained by the method of section 4; we have always to distinguish the cases $\left(\frac{f}{p}\right) = \pm 1$, and for the smaller p it proves necessary to consider separately the various possibilities for $h(f)$. There are regular forms, shown in Tables II, III, which show that these bounds are best possible.

11. Proof of Theorem 5. We assume

$$(18) \qquad \omega(f) \equiv 0 \pmod{p^2},$$

for some odd p, and also (17), and we have to show that f must be irregular if $p \geqslant 5$. For $p = 3$ this is not true without some further hypothesis, but if we assume that $\omega(f)$ divides by 27, or $d(f)$ by 3^6 (the latter of which is implied by the former), then we can prove f irregular, and this gives us what is wanted; for the cases with $\omega(f)$ not divisible by 9 have been dealt with in Theorems 3 and 4.

The main difficulty is to find some bound for p; for the application of Lemma 2, case 1, may lead to a form with d no longer divisible by p. The required bound is given by the following three lemmas, and we omit for brevity the routine calculations needed to complete the proof.

LEMMA 7. Assume (17) and (18); then for $p \geqslant 5$, f is irregular if $\left(\frac{f}{p}\right) = +1$.

Proof. Assuming f to be regular, it must represent 1, so, working with a reduced f, we have $a = 1$.

First suppose $p \geqslant 13$. Then there exist $q' < q'' \leqslant 11 < p$, such that neither q' nor q'' divides $d_o(f)$; since Table I shows that $d_o(f)$ divides by at most three primes. Then by (17) neither of q', q'' divides $d(f)$, so f has neither q'- nor q''-progressions. Therefore f if regular must represent q' or q'', if either of them is a quadratic residue (mod p); and if not, then f must represent $q'q''$. But this gives us, by $(13)_2$, $b \leqslant q'q''$, whence $C \leqslant 4q'q'' = 308$. Now by (4) and (18) $C \geqslant 3p^2 \geqslant 507$, which is a contradiction.

In case $p = 11$, f must represent 37, if regular, since $\left(\frac{37}{11}\right) = 1$ and $h(f)$, and so f, can not have 37-progressions. Similarly, if $p = 7$, f represents 29. Thus in these two cases we have $b \leqslant 37, 29$, $C \leqslant 148, 116$, $C < 3p^2$, and again we have a contradiction.

For $p = 5$, we have $b \leqslant 11$, unless $h(f)$ is h_9, in which case alone 11-progressions exist; in this case $b \leqslant 6$. Hence the result again follows.

LEMMA 8. With the hypotheses of Lemma 7, except that $\left(\frac{f}{p}\right) = -1$, let $P = P(p)$ be the least prime non-residue of p; then f is irregular if $P > 17$.

Proof. Suppose $P \geqslant 19$, f is regular, and Q is the next least prime non-residue (mod p). Then f can have neither P- nor Q-progressions, and so must represent both P and Q. This gives $a \leqslant P$, $b \leqslant Q$, $C \leqslant 4PQ$. We have a contradiction with $C \geqslant 3p^2$ if we can prove

$$P < 3p/4, \quad Q < p.$$

But these inequalities hold, since if not, the $\frac{1}{2}(p - 1)$ non-residues of p lying between 1 and $p - 1$ are either all multiples of P, or all exceed $3p/4$.

LEMMA 9. With the hypotheses of Lemma 8, f is irregular if $P \leqslant 17$ and $p > 23$.

Proof. First suppose $P(p) = 2$ and $p > 29$. Then one of each of the pairs of integers 19, 38; 23, 46; 29, 58 is a quadratic non-residue of p. Hence, since $h(f)$ and f can have no 19-, 23- or 29-progressions, while at most one of these six integers can be $\equiv -d \pmod{16}$, two of them must be represented by f. This gives $a \leqslant 46$, $b \leqslant 58$,

$$p^2 \leqslant C/3 \leqslant 4ab/3 \leqslant 4.46.58/3 < 60^2, \quad p < 60,$$

if f is regular.

If f is regular and $p > 31$, $P = 3$, f represents either 19 or 57, and either 31 or 93, unless $d(f) \equiv 3 \pmod 9$, in which case we use instead the four numbers 23, 69, 29, 87. This gives us $p < 90$.

Similar arguments in the cases $P = 5, \ldots, 17$ give bounds for p in no case exceeding 560; and we note that if $P > 3$, $p \equiv \pm 1 \pmod{24}$.

So we may suppose $29 \leqslant p < 560$, $p \equiv \pm 1 \pmod{24}$ if $p > 90$. It is not difficult to verify that for such p we can satisfy

$$19 \leq q' < q'', \quad q'q'' < \tfrac{1}{4}p^2, \quad \left(\frac{q'}{p}\right) = \left(\frac{q''}{p}\right) = -1,$$

whence the result by the method of Lemma 8.

12. <u>Proof of Theorem 6.</u> We investigate separately the four cases possible for a classic form by Lemma 1(b)(ii) We have in these cases, for a reduced f,

$$a, b \leqq 1, 41; \; 11, 19; \; 13, 29; \; 23, 31,$$

on noting that 3- and 11-, 5- and 13-progressions do not both exist. In each of the rather numerous cases we have to seek a number n'' with the properties mentioned in section 4(iv). I omit the details.

13. <u>Regularity of the forms</u> $h_1, \ldots, h_{24}$. General methods of proving the regularity of a form have been discussed by Jones and Pall (loc. cit.), and they have proved the regularity of our form h_9. The regularity of all but three of the other forms in Table I follows from the fact that they represent genera each of a single class. These three (and h_9) belong to two-class genera, and I prove them regular by showing that each represents every positive integer represented by any form in its companion class.

<u>Regularity of</u> h_{13}. Write $h = h_{13}$ and denote by h' a representative of the companion class. Then

$$h = x^2 + 2y^2 + yz + 2z^2, \qquad (15)$$

$$h' = x^2 + xy + y^2 + 5z^2.$$

A solution of $h' = n$ gives one of

$$u^2 + 3v^2 + 5z^2 = n,$$

with $u, v = x + \frac{1}{2}y, \frac{1}{2}y;\ y + \frac{1}{2}x, \frac{1}{2}x;$ or $\frac{1}{2}(x - y), \frac{1}{2}(x+y),$ according as y, x or neither is even. Then we have

$$n = h(u, z - v, z + v).$$

<u>Regularity of h_{14}</u>. With a similar notation,

(17)

$$h = x^2 + xy + 2y^2 + 2yz + 3z^2,$$
$$h' = x^2 + xy + y^2 + yz + 6z^2.$$

A solution of $h' = n$ gives

$$12n = u^2 + 3v^2 + 68z^2, \quad u = 3y + 2z,\ v = 2x + y.$$

Now $u \equiv v \pmod 2$, and if u, v are odd, we may replace them, without altering the value of $u^2 + 3v^2$, by $\frac{1}{2}(u \pm 3v)$, $\frac{1}{2}(u \mp v)$, which with proper sign are both even. We can therefore solve

$$3n = X^2 + 3Y^2 + 17z^2.$$

Now we may suppose, by putting if necessary $-X$ for X, that $X \equiv -z \pmod 3$, so that $w = (X - 2z)/3$ is an integer. Then we find

$$n = h(Y - z, 2z, w).$$

<u>Regularity of h_{15}</u>. We have

$$h = x^2 + xy + 2y^2 + 3z^2,$$
$$h' = x^2 + xy + y^2 + 7z^2.$$

As in the case of h_{13}, a solution of $h' = n$ leads to one of

$$n = u^2 + 3v^2 + 7z^2 \quad = h(u + z, -2z, v).$$

14. <u>Regular forms with d not square-free, but divisible by neither 4 nor 9.</u> A representative of every possible class of such forms is shown in Table II. With the help of Theorems 2 to 5, simple calculations show that these 34 forms are the only possibilities. 29 of these belong to single-class genera and so are regular. I discuss the others briefly.

f_{19} is dealt with in the same way as h_{15}. In fact $g_7(f_{19})$ is h_{15}.

It can easily be proved that in case $\omega(f)$ and $d(f)$ are divisible precisely by the first, second power of p respectively, so that case 2 of Lemma 2 is applicable, while $g_p(f)$ has p-progressions, then the regularity of $g_p(f)$ is a sufficient as well as necessary condition for that of f. This result (which holds also for $p = 2$) establishes the regularity of f_{10}, f_{17} and f_{18}.

The remaining case is f_1. f_1 has to represent all n such that

$$n \not\equiv 14 \cdot 4^k \pmod{4^{k+2}}, \quad n \equiv 0 \pmod{25} \text{ or } \not\equiv 0 \pmod 5.$$

For any such n, $2n$ is the sum of three squares, so we have

$$2n = u^2 + v^2 + w^2$$

and we may suppose $u \equiv v \pmod 5$; for if this cannot be satisfied by permuting u, v, w and changing their signs, we have $u^2, v^2, w^2 \equiv 0, 1, 4 \pmod 5$ in some order, and $n \equiv 0 \pmod 5$. But then we must have $n \equiv 0 \pmod{25}$, and so we can have $u \equiv v \equiv w \equiv 0 \pmod 5$.

So we have

$$4n = X^2 + 2Y^2 + 25Z^2,$$

with $X = u + v$, $Y = w$, $Z = (u - v)/5$. Since $u + v + w$ must be even, we have $X \equiv Y \equiv Z \pmod 2$. So we can satisfy

$$4n = (2x + z)^2 + 2(2y + z)^2 + 25z^2 = 4f_1(x, y, z).$$

15. Regular forms with d divisible by 9 but not by 4. All possible classes of this type are represented in Table III, so that Tables I to III together show all possibilities with $d \not\equiv 0 \pmod 4$. There are just two, marked ?, whose regularity I have been unable to prove. The proofs are by the methods outlined in the preceding two sections.
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16. The method can be extended to four or more variables, or to forms which are regular with one exception; but in either case the calculations are formidable. I am indebted to Professor Davenport for reading an earlier draft of this paper, and making a number of suggestions.

Appendix.

Table I; representative forms of the twenty-four regular classes with square-free d.

i	h_i	$d(h_i)$	Primes for which h_i has progressions.
1	$x^2 + xy + y^2 + yz + z^2$	2	2
2	$x^2 + xy + y^2 + z^2$	3	3
3	$x^2 + xy + y^2 + yz + 2z^2$	5	5
4	$x^2 + xy + y^2 + 2z^2$	6	2
5	$x^2 + y^2 + xz + yz + 2z^2$	6	3
6	$x^2 + y^2 + yz + 2z^2$	7	7
7	$x^2 + y^2 + xz + yz + 3z^2$	10	2
8	$x^2 + xy + 2y^2 + 2yz + 2z^2$	10	5
9	$x^2 + y^2 + yz + 3z^2$	11	11
10	$x^2 + xy + 2y^2 + yz + 2z^2$	13	13
11	$x^2 + xy + y^2 + yz + 5z^2$	14	2
12	$x^2 + y^2 + yz + 4z^2$	15	3
13	$x^2 + 2y^2 + yz + 2z^2$	15	5
14	$x^2 + xy + 2y^2 + 2yz + 3z^2$	17	17
15	$x^2 + xy + 2y^2 + 3z^2$	21	3
16	$x^2 + 2y^2 + xz + yz + 3z^2$	21	7
17	$x^2 + 2y^2 + xz + 3z^2$	22	2
18	$x^2 + xy + y^2 + 10z^2$	30	2, 3, 5
19	$x^2 + xy + 3y^2 + xz + 3z^2$	30	2
20	$x^2 + xy + 2y^2 + xz + 5z^2$	33	3
21	$x^2 + y^2 + xz + yz + 11z^2$	42	2, 3, 7
22	$x^2 + xy + 3y^2 + 3yz + 5z^2$	46	2

Table I (contd)

i	h_i	$d(h_i)$	Primes for which h_i has progressions.
23	$x^2 + 2y^2 + xz + 9z^2$	70	2, 5, 7
24	$x^2 + xy + 5y^2 + 5yz + 6z^2$	78	2, 3, 13

Table II. Representatives of all classes of regular forms with d not square-free, but divisible by neither 4 nor 9.

i	f_i II	ω	d	Related forms	
1	$x^2 + 2y^2 + xz + 2yz + 7z^2$	1	50	h_1	
2	$2x^2 + xy + 2y^2 + 5yz + 5z^2$	5	25	h_3	
3	$x^2 + xy + 4y^2 + 5yz + 10z^2$	5	125	f_2, h_3	
4	$2x^2 + 2xy + 3y^2 + xz + 3yz + 7z^2$	5	125	f_2, h_3	
5	$x^2 + xy + 2y^2 + 7z^2$	7	49	h_6	
6	$2x^2 + 2xy + 3y^2 + 5yz + 5z^2$	5	50	h_7	
7	$x^2 + xy + 9y^2 + 10yz + 10z^2$	5	250	f_6, h_7	
8	$2x^2 + 2xy + 3y^2 + 5yz + 15z^2$	5	250	f_6, h_7	
9	$x^2 + xy + 4y^2 + 5yz + 5z^2$	5	50	h_8	
10	$x^2 + xy + 3y^2 + 11z^2$	11	121	h_9	Lemma 2
11	$2x^2 + xy + 5y^2 + 13yz + 13z^2$	13	169	h_{10}	
12	$3x^2 + xy + 3y^2 + 7yz + 7z^2$	7	98	h_{11}	
13	$x^2 + xy + 9y^2 + 7yz + 21z^2$	7	686	f_{12}, h_{11}	
14	$3x^2 + 2xy + 5y^2 + 3xz + yz + 13z^2$	7	686	f_{12}, h_{11}	
15	$2x^2 + xy + 2y^2 + 5z^2$	5	75	h_{12}	
16	$3x^2 + xy + 7y^2 + 15yz + 15z^2$	5	375	f_{15}, h_{12}	
17	$x^2 + xy + 4y^2 + 5z^2$	5	75	h_{13}	Lemma 2
18	$3x^2 + 3xy + 5y^2 + 2xz + yz + 6z^2$	17	289	h_{14}	Lemma 2

Table II, contd. d 4∤d; 9∤d.

i	f_i	ω	d	Related forms.
19	$x^2 + xy + 2y^2 + 21z^2$	7	147	h_{15}
20	$3x^2 + xy + 3y^2 + 2xz - 2yz + 5z^2$	7	147	h_{16}
21	$x^2 + xy + 3y^2 + 22z^2$	11	242	h_{17}
22	$2x^2 + 5y^2 + 5yz + 5z^2$	5	150	h_{18}
23	$x^2 + 10y^2 + xz + 19z^2$	5	750	f_{22}, h_{18}
24	$3x^2 + 3xy + 7y^2 + 10z^2$	5	750	f_{22}, h_{18}
25	$x^2 + 5y^2 + xz + 5yz + 9z^2$	5	150	h_{19}
26	$3x^2 + xy + 3y^2 + 2xz + 2yz + 22z^2$	5	750	f_{25}, h_{19}
27	$2x^2 + xy + 7y^2 + xz + 3yz + 7z^2$	11	363	h_{20}
28	$5x^2 + 4xy + 5y^2 + 3xz - 3yz + 5z^2$	7	294	h_{21}
29	$5x^2 + 2xy + 10y^2 + 49xz + 133z^2$	7	2058	f_{28}, h_{21}
30	$5x^2 + 5xy + 7y^2 + 46xz + 138z^2$	23	1058	h_{22}
31	$3x^2 + xy + 3y^2 + 10z^2$	5	350	h_{23}
32	$3x^2 + xy + 3y^2 + 14z^2$	7	490	h_{23}
33	$x^2 + xy + 9y^2 + 70z^2$	35	2450	f_{31}, f_{32}, h_{23}
34	$x^2 + 13y^2 + 13xz + 13yz + 65z^2$	13	1014	h_{24}

The last column shows the forms derivable from each f_i by the processes of Lemma 2.

Table III. Representatives of all possible regular classes with d divisible by 9 but not by 4.

~~i Coefficients of f_i (a, b, c; r, s, t) d Related forms~~

~~(a) ≢ 0 (mod 3).~~

~~35~~

Table III d divisible by 9 but not by 4.

	i	Coefficients of f_i (a, b, c; r, s, t)	Classes in genus	ω	d	Related forms
$\omega \not\equiv 0 \pmod 3$	35	1, 1, 5; 1, 1, 0	1	1	18	h_1
	36	1, 2, 3; 2, 1, 0	1	1	18	h_1
	37	1, 1, 7; 0, 1, 0	2	1	27	h_2
	38	1, 2, 4; 1, 0, 1		1	27	h_2
	39	1, 2, 7; 2, 0, 1	1	1	45	h_3
	40	1, 5, 10; 8, 0, 1	1	1	126	h_{11}
	41	2, 5, 7; 5, 1, 0	1	5	225	f_2, f_{39}, h_3
	42	5, 5, 77; 35, 0, 3	1	7	882	f_{12}, f_{40}, h_{11}
	43	1, 10, 29; 5, 1, 0	1 ?	5	1125	$f_{2,3,39,41}$, h_3
	44	2, 7, 35; 20, 0, 1	2	5	1125	$f_{2,4,39,41}$, h_3
$\omega \equiv 0 \pmod 9$	45	1, 7, 9; 9, 0, 1	1	9	162	h_1
	46	2, 5, 9; 9, 0, 2	1	9	162	h_1
	47	1, 7, 9; 0, 0, 1	2	9	243	h_2
	48	2, 5, 8; 5, 1, 2	1	9	243	h_2
	49	2, 8, 9; 9, 0, 1	1	9	405	h_3
	50	1, 7, 18; 0, 0, 1	2	9	486	h_4
	51	5, 9, 9; 9, 0, 3	1	9	1134	h_{11}
	52	9, 9, 11; 6, 3, 9	1	9	2430	h_{18}
	53	7, 7, 90; 0, 45, 4	1	45	2025	f_2, f_{49}, h_3
	54	5, 17, 1764; 0, 126, 5	1	63	7938	f_{12}, f_{51}, h_{11}
	55	7, 13, 37; 13, 1, 2	1	45	12150	f_{22}, f_{52}, h_{18}
$\omega \equiv \pm 3 \pmod 9$, $\lambda \equiv \pm 4 \pmod{27}$	56	1, 1, 3; 0, 0, 1	1	3	9	h_2
	57	1, 1, 6; 0, 0, 1	1	3	18	h_4
	58	2, 2, 2; -1, 1, 2	1	3	18	h_5
	59	2, 2, 3; 0, 0, 1	1	3	45	h_{12}

Table III contd.

	i	Coefficients of f_i (a, b, c; r, s, t)	Classes in genus	ω	d	Related forms
$d \equiv \pm 9 \pmod{27}$	60	1, 3, 4; 0, 1, 0	2	3	45	h_{13}
	61	1, 3, 6; 3, 0, 0	2	3	63	h_{15}
	62	2, 2, 9; 6, 0, 1	1	3	63	h_{16}
	63	1, 1, 30; 0, 0, 1	1	3	90	h_{18}
	64	2, 3, 5; 3, 2, 0	1	3	90	h_{19}
	65	2, 3, 5; 3, 1, 0	1	3	99	h_{20}
	66	3, 3, 5; 3, 3, 0	1	3	126	h_{21}
	67	2, 3, 11; 3, 2, 0	1	3	234	h_{24}
	68	1, 4, 15; 0, 0, 1	1	15	225	f_{15}, f_{59}, h_{12}
	69	5, 6, 11; 3, 5, 0	1	15	1125	$f_{15,16,59,68}$, h_{12}
	70	2, 2, 15; 0, 0, 1	2	15	225	f_{17}, f_{60}, h_{13}
	71	3, 6, 7; 0, 0, 3	2	21	441	f_{19}, f_{61}, h_{15}
	72	2, 8, 21; 21, 0, 1	1	21	441	f_{20}, f_{62}, h_{16}
$\omega \equiv \pm 3 \pmod 9$	73	5, 5, 6; 0, 0, 5	1	15	450	f_{22}, f_{63}, h_{18}
	74	3, 7, 30; 0, 0, 3	1	15	2250	$f_{22,23,63,73}$, h_{18}
	75	1, 19, 30; 0, 0, 1	1	15	2250	$f_{22,24,63,73}$, h_{18}
	76	3, 7, 10; 10, 0, 3	1	15	450	f_{25}, f_{64}, h_{19}
	77	9, 9, 65; 45, 0, 3	1	15	2250	$f_{25,26,64,76}$, h_{19}
	78	6, 7, 66; 0, 33, 6	1	33	1089	f_{27}, f_{65}, h_{20}
	79	2, 11, 21; 21, 0, 2	1	21	882	f_{28}, f_{66}, h_{21}
	80	11, 11, 15; 3, 3, 8	1	21	6174	$f_{28,29,66,79}$, h_{21}
	81	3, 17, 39; 39, 0, 3	1	39	3042	f_{34}, f_{67}, h_{24}
$\omega \equiv \pm 3 \pmod 9$, $d \equiv \pm 27 \pmod{81}$	82	1, 1, 9; 0, 0, 1	2 (82, 83)	3	27	f_{56}, h_2 (82, 83)
	83	1, 3, 3; 3, 0, 0		3	27	
	84	2, 2, 2; 1, 1, 1	1	3	27	f_{56}, h_2

$\omega \equiv \pm 3 \pmod{9}$, $d \equiv \pm 27 \pmod{81}$

i	Coefficients of f_i (a, b, c; r, s, t)	Classes in genus	ω	d	Related forms
85	1, 1, 18; 0, 0, 1	1	3	54	f_{57}, h_4
86	1, 4, 6; 6, 0, 1	1	3	54	f_{58}, h_5
87	2, 3, 3; 3, 0, 0	1	3	54	f_{57}, h_4
88	2, 2, 5; -1, 1, 2	1	3	54	f_{58}, h_5
89	2, 5, 5; 5, 1, 2	1	3	135	f_{59}, h_{12}
90	1, 3, 12; 3, 0, 0	2	3	135	f_{60}, h_{13}
91	2, 2, 9; 0, 0, 1	2	3	135	f_{60}, h_{13}
92	2, 3, 8; 0, 1, 0	2	3	189	f_{61}, h_{15}
93	1, 4, 15; 6, 0, 1	1	3	189	f_{62}, h_{16}
94	2, 5, 9; 9, 0, 1	1	3	189	f_{62}, h_{16}
95	3, 3, 11; 3, 0, 3	1	3	270	f_{63}, h_{18}
96	1, 6, 13; 6, 1, 0	1	3	270	f_{64}, h_{19}
97	5, 5, 6; 6, 0, 5	1	3	270	f_{64}, h_{19}
?98	1, 6, 13; 6, 1, 0	2	3	297	f_{65}, h_{20}
99	2, 5, 15; 12, 0, 1	1	3	297	f_{65}, h_{20}
100	1, 9, 13; 9, 1, 0	1	3	378	f_{66}, h_{21}
101	2, 5, 15; 9, 0, 2	1	3	378	f_{66}, h_{21}
102	5, 5, 9; 3, 3, 4	1	3	702	f_{67}, h_{24}
103	3, 7, 10; 5, 0, 3	1	15	675	$f_{15,59,68,89}$, h_{12}
?104	2, 15; 32; 15, 1, 0	2	15	3375	$f_{15,16,59,68,69,89,103}$, h_{12}
105	1, 4, 45; 0, 0, 1	2	15	675	$f_{17,60,70,91}$, h_{13}
106	5, 6, 6; 3, 0, 0	2	15	675	$f_{17,60,70,90}$, h_{13}
107	2, 8, 21; 0, 0, 1	2	21	1323	$f_{19,61,71,92}$, h_{15}
108	6, 7, 10; 7, 3, 0	1	21	1323	$f_{20,62,72,93}$, h_{16}
109	5, 5, 42; 21, 0, 4	1	21	1323	$f_{20,62,72,94}$, h_{16}
110	7, 7, 15; 15, 0, 1	1	15	1350	$f_{22,63,73,95}$, h_{18}

Table III contd

	i	Coefficients of f_i (a, b, c; r, s, t)	Classes in genus	ω	d	Related forms
ω ≡ ±3 (mod 9), α ≡ ±27 (mod 81)	111	5, 9, 11; 9, 5, 0	1	15	1350	$f_{25,64,76,96}$, h_{19}
	112	1, 19, 45; 45, 0, 1	1	15	1350	$f_{25,64,76,97}$, h_{19}
	113	9, 11, 21; 3, 9, 6	1	15	6750	$f_{21,23,63,73,74,95,110}$, h_{18}
	114	3, 17, 35; 5, 0, 3	1	15	6750	$f_{22,24,63,73,75,95,110}$, h_{18}
	115	5, 17, 33; 27, 0, 5	1	15	6750	$f_{25,26,64,76,77,97,112}$, h_{19}
	116	7, 13, 105; 0, 45, 7	1	15	6750	$f_{25,26,64,76,77,96,111}$, h_{19}
	117	7, 10, 13; 5, 1, 4	1	33	3267	$f_{27,65,78,99}$, h_{20}
	118	6, 7, 19; 7, 6, 0	1	21	2646	$f_{28,66,79,100}$, h_{21}
	119	5, 6, 35; 3, 7, 6	1	21	2646	$f_{28,66,79,101}$, h_{21}
	120	9, 14, 429; 117, 0, 6	1	39	9126	$f_{34,67,81,102}$, h_{24}
ω ≡ ±3 (mod 9), α ≡ ±81 (mod 243)	121	1, 3, 7; 0, 1, 0	3	3	81	$f_{83,56}$, h_2
	122	1, 4, 6; 3, 0, 1	1	3	81	$f_{84,56}$, h_2
	123	1, 6, 7; 0, 1, 0	1	3	162	$f_{87,57}$, h_4
	124	2, 2, 12; 3, 0, 1	2 (124–125)	3	162	$f_{86,58}$, h_5
	125	2, 3, 8; 3, 2, 0		3	162	$f_{86,58}$, h_5
	126	2, 5, 6; 6, 0, 1	2	3	162	$f_{88,58}$, h_5
	127	1, 7, 33; 9, 0, 1	1	3	810	$f_{95,63}$, h_{18}
	128	5, 11, 21; 3, 0, 5	1	15	4050	$f_{127,110,95,63}$, h_{18}
ω = 3, α ≡ 0 (mod 243)	129	2, 3, 11; 3, 1, 0	3	3	243	$f_{120,84,56}$, h_2
	130	2, 5, 24; 15, 0, 1	4	3	486	$f_{124,88,58}$, h_5