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# Ternary positive definite quadratic forms are determined by their theta series

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## 1 Introduction

In this paper we are always dealing with quadratic forms with real coefficients but it wouldn't make any difference to consider only rational forms. The notion of equivalence is that over $\mathbb{Z}$, i.e. for $n$-dimensional forms we define $f \sim_{\mathbb{Z}} g \iff \exists\, T \in \mathbf{GL}_n(\mathbb{Z}) : f(T\mathbf{x}) = g(\mathbf{x})$. When we refer to the coefficients of a form $f$ we refer to its Gram matrix i.e. $f(\mathbf{x}) = \sum_i f_{ii} x_i^2 + 2 \sum_{i<j} f_{ij} x_i x_j$. The $n$-dimensional forms are embedded into $\mathbb{R}^{n(n+1)/2}$ by the coefficients $f_{ij}$ with $i \leq j$.

**Definition 1.1** *The representation number $A_X(f,t)$ of a real number $t \in \mathbb{R}_0^+$ by an $n$-dimensional positive definite form $f$ with respect to a set $X \subset \mathbb{Z}^n$ is*

$$\begin{aligned} A_X(f,t) &= \sharp\{\mathbf{x} \in X : f(\mathbf{x}) = t\} \\ A(f,t) &= A_{\mathbb{Z}^n}(f,t) \end{aligned}$$

It is well known that binary positive definite forms are determined up to integral equivalence by their series of representation numbers (the theta series). We know even more: Up to a factor and order there is only one pair of classes of inequivalent positive definite binary forms with the same set of represented numbers (without multiplicities). See [Wat] for a proof and reference to older results. In higher dimensions there are counterexamples. E.Witt proved that there are two classes of positive definite even unimodular forms of dimension 16 which have the same representation numbers [Wit]. Since then such examples were found in dimensions 12, 8 and 4 ([Kne],[Kit],[Sch1]). Conway and Sloane [Con] found a 4–parameter family of pairs of quaternary forms with the theta series of both forms being equal. Examples suggest that the forms of these pairs are inequivalent (at least if some inequalities for the parameters hold), but this is not known in general.

In her thesis [Suw] Suwa-Bier shows that there are at most 4 pairwise inequivalent positive definite ternary forms belonging to a given theta series. The proof is divided into many distinct cases in some way similar to the calculations presented here. She gives also a short proof for “at most 30”.

Our aim is to find a bound $b(f)$ with

$$A(f,t) = A(g,t)\ \forall t \le b(f) \implies f \sim_{\mathbb{Z}} g \tag{1}$$

where $f, g$ are ternary positive definite forms. For such forms there is a well known fundamental domain, the Seeber-Eisenstein reduced forms, containing exactly one representative of each class (see [Eis] or [O'M] p.142f). The set of Eisenstein reduced forms in $\mathbb{R}^6$ is the union of two convex cones. We choose other representatives by a piecewise linear map that doesn't change the classes and defines a bijection to one convex cone in $\mathbb{R}^6$ which we denote by $V$. In Definition 2.1 we give the linear conditions which define this fundamental domain $V$. By restricting ourselves to reduced forms we can replace the notion of integral equivalence by equality. To exploit the condition $A(f,t) = A(g,t) \forall\, t \in \mathbb{R}_0^+$ on $(f,g) \in V \times V$ we divide $V$ into domains of forms with finitely many prescribed successively minimal vectors. It is sufficient to consider only representation numbers with respect to $\mathbb{Z}_*^3$ (by which we denote the set of primitive vectors in $\mathbb{Z}^3$ with the last nonzero coefficient being positive). Let $\mathbf{x}_1, \ldots, \mathbf{x}_j \in \mathbb{Z}_*^3$ be the first $j$ successively minimal vectors of $f$, i.e. $f(\mathbf{x}_i) = \min f\left(\mathbb{Z}_*^3 \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}\}\right)$, and $\mathbf{y}_1, \ldots, \mathbf{y}_j$ those of $g$. The equality of the representation numbers leads to $f(\mathbf{x}_i) = g(\mathbf{y}_i)$. These are linear equations with integral coefficients in the variables $f_{ij}, g_{ij}$.

In Sect. 2 we describe the subdivision of $V$ into domains with finitely many prescribed successively minimal vectors. The main problem is to describe the refinement of such a subdivision to one with one more prescribed minimal vector in each member. In any case we give a finite set of vectors sufficient to be taken into account as the next minimal ones.

**Definition 1.2**

$$\begin{aligned} \Delta &= \{(f,f) \in \overline{V} \times \overline{V}\} \\ D &= \{(f,g) \in V \times V \ :\ A(f,t) = A(g,t) \forall\, t \in \mathbb{R}_0^+\} \end{aligned}$$

*where $\overline{V}$ denotes the closure of $V \subset \mathbb{R}^6$.*

In Sect. 3 we define a decreasing series $(\mathfrak{T}_i)$ of coverings of $D$ that can be explicitly computed, each refining its predecessor. The member sets of these coverings contain pairs of forms with prescribed minimal vectors and the refinement is done, roughly said, by choosing the next minimal vectors. These coverings approximate $D$ in the sense that $D = \bigcap_i \bigcup_{T \in \mathfrak{T}_i} T \cap (V \times V)$. By explicitly computing them with a computer it turns out that after finitely many steps each member set is contained in the diagonal $\Delta$. To achieve this, we must take care of the following situation: There are pairs of forms with any given number of the first successively minimal vectors lying in $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle_{\mathbb{Z}}$ (the 2-dimensional sublattice spanned by the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$). Involving only minimal vectors from this sublattice the equality of the theta series only implies $f_{|\langle \mathbf{e}_1, \mathbf{e}_2 \rangle_{\mathbb{Z}}} = g_{|\langle \mathbf{e}_1, \mathbf{e}_2 \rangle_{\mathbb{Z}}}$ after a limited number of steps, but not $f = g$. Thus any $T \in \mathfrak{T}_i$ with $f_{|\langle \mathbf{e}_1, \mathbf{e}_2 \rangle_{\mathbb{Z}}} = g_{|\langle \mathbf{e}_1, \mathbf{e}_2 \rangle_{\mathbb{Z}}} \ \forall (f, g) \in T$ is refined by considering successively minimal vectors only from $\mathbb{Z}^3_* \setminus \langle \mathbf{e}_1, \mathbf{e}_2 \rangle_{\mathbb{Z}}$. We still get equations from the minimal vectors because $(f, g) \in D$ and $f_{|\langle \mathbf{e}_1, \mathbf{e}_2 \rangle_{\mathbb{Z}}} = g_{|\langle \mathbf{e}_1, \mathbf{e}_2 \rangle_{\mathbb{Z}}}$ implies $A_{\mathbb{Z}^3_* \setminus \langle \mathbf{e}_1, \mathbf{e}_2 \rangle_{\mathbb{Z}}}(f, t) = A_{\mathbb{Z}^3_* \setminus \langle \mathbf{e}_1, \mathbf{e}_2 \rangle_{\mathbb{Z}}}(g, t)$. The sublattice $\langle \mathbf{e}_1, \mathbf{e}_3 \rangle_{\mathbb{Z}}$ is treated the same way.

An explicit bound $b(f)$ for which (1) holds is given in Theorem 4.4 and 4.5. In the case of forms with integral coefficients the theory of modular forms gives a bound for the number of coefficients of a theta series which determine that series among all theta series belonging to forms with the same level. Theorem 4.4 also gives a bound for the coefficients that determine the theta series of an arbitrary positive definite ternary form with real coefficients among all other theta series belonging to ternary forms (possibly with other determinant), thus including the case of theta series that are not modular forms.

With the constructive subdivision of $V$ into domains of forms with prescribed successively minimal vectors we can also answer the question, which ternary positive definite forms have given representation numbers in an interval $[0, t]$.

## 2 Successively minimal vectors

First we state how to identify the class of a form from its Gram matrix. We recall that a positive definite quadratic form $f$ in $n$ variables is Minkowski reduced iff

$$\forall k = 1 \dots n \ \forall \, \mathbf{x} \in \mathbb{Z}^n \left( \gcd(x_k, \dots, x_n) = 1 \implies f(\mathbf{x}) \geq f_{kk} \right) \tag{2}$$

For forms of dimension $n \leq 4$ we can replace in (2) $\forall \, \mathbf{x} \in \mathbb{Z}^n$ by $\forall \, \mathbf{x} \in \{-1, 0, 1\}^n$ (see [Min]p.78). We obtain the following reduction conditions from Eisenstein's by a piecewise linear map that folds up the two convex cones of the Eisenstein reduced forms to one convex cone which is computationally easier to handle. This map is bijective and does not affect the classes. Thus in every class of ternary positive definite forms there is exactly one representative satisfying Definition 2.1 as well as exactly one Eisenstein reduced form. For the original Eisenstein conditions see [Eis], [O'M] p.142f or [Sch2] where the piecewise linear map and a reduction procedure is described.

**Definition 2.1** *We say that a positive definite ternary form $f$ is reduced iff the following conditions are satisfied.*

1. *$f$ is Minkowski reduced*
2. $f_{12} \geq 0$ , $f_{13} \geq 0$ *and* $(f_{12} = 0 \vee f_{13} = 0) \implies f_{23} \geq 0$
3. $f_{11} = f_{22} \implies |f_{23}| \leq f_{13}$
   $f_{22} = f_{33} \implies f_{13} \leq f_{12}$

*4.* $f_{11} + f_{22} - 2f_{12} - 2f_{13} + 2f_{23} = 0 \implies f_{11} - 2f_{13} - f_{12} \leq 0$
$2f_{12} = f_{11} \implies f_{13} \leq 2f_{23}$
$2f_{13} = f_{11} \implies f_{12} \leq 2f_{23}$
$2f_{23} = f_{22} \implies f_{12} \leq 2f_{13}$
$2f_{23} > -f_{22}$

*The set of reduced forms is denoted by $V$.*

The closure of $V \subset \mathbb{R}^6$ is a convex cone spanned by the 11 forms from the sets $M_1$ and $M_2$, i.e. $\overline{V} = \{\sum \lambda_i f_i \ : \ f_i \in M_1 \cup M_2, \lambda_i \in \mathbb{R}_0^+\}$, where

$$M_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix} \right\}$$

$$M_2 = \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & \pm 1 \\ 1 & \pm 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \right\}$$

**Lemma 2.2** *Let $f \in M_2$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$. Then $f(\mathbf{x}) \geq \|\mathbf{x}\|_\infty^2$, where $\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|, |x_3|\}$.*

*Proof.* Since $f \in M_2$ we have $f_{ii} = 2$ and $f_{ij} \in \{-1, 0, 1\}$ for $i \neq j$. Let $\{i, j, k\} = \{1, 2, 3\}$ such that $|x_i| = \max\{|x_1|, |x_2|, |x_3|\}$. Then $|x_j x_k| = \min_{\nu<\mu}\{|x_\nu x_\mu|\}$. Now for $f \in M_2$ either one of $f_{12}, f_{13}, f_{23}$ is 0 or $(f_{12}, f_{13}, f_{23}) = (1, 1, 1)$. In both cases at most two of the terms $f_{12}x_1x_2$, $f_{13}x_1x_3$, $f_{23}x_2x_3$ can be negative and their sum has the lower bound $-|x_i x_k| - |x_i x_j|$. Now

$$\begin{aligned} f(\mathbf{x}) &= \sum_\nu f_{\nu\nu} x_\nu^2 + 2 \sum_{\nu<\mu} f_{\nu\mu} x_\nu x_\mu \ \geq \ \sum_\nu 2x_\nu^2 - 2|x_i x_k| - 2|x_i x_j| \\ &= 2(1/2\ |x_i| - |x_k|)^2 + 2(1/2\ |x_i| - |x_j|)^2 + x_i^2 \ \geq \ \|\mathbf{x}\|_\infty^2 \end{aligned}$$

□

**Definition 2.3** *Let $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^3$, $X, Y \subset \mathbb{Z}^3$. Let*

$$\begin{aligned} \mathbf{x} \preceq \mathbf{y} \quad (\textit{resp. } \mathbf{y} \succeq \mathbf{x}) \quad &:\Longleftrightarrow \quad \forall f \in V \ : \ f(\mathbf{x}) \leq f(\mathbf{y}) \\ X^{\not\succeq Y} \ &= \ \{\mathbf{x} \in X \ : \quad \forall \mathbf{y} \in Y \ : \ \mathbf{x} \not\succeq \mathbf{y}\} \\ \mathrm{MIN}(X) \ &= \ \{\mathbf{x} \in X \ : \quad \forall \mathbf{y} \in X \ : \ (\mathbf{y} \preceq \mathbf{x} \implies \mathbf{y} = \mathbf{x})\} \end{aligned}$$

$\mathrm{MIN}(X)$ is the set of minimal elements of $X$ related to the order relation $\preceq$. Every sequence $\mathbf{x}_1 \succeq \mathbf{x}_2 \succeq \ldots$ is eventually constant. Thus every nonempty set has minimal elements. For every $X \subset \mathbb{Z}^3$ the set $\mathrm{MIN}(X)$ is finite and can be determined effectively (this is true for the analogous definition for arbitrary dimensions with the notion of Minkowski reduction as well). For special $X \subset \mathbb{Z}^3$ we make this explicit giving an a priori bound for $\mathrm{MIN}(X)$.

**Lemma 2.4** *1. Let $a \in \mathbb{Z}$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$ with $(x_2, x_3) \neq (0, 0)$. Then*
$\mathbf{x} \not\succeq (a, 1, 0) \implies \|\mathbf{x}\|_\infty < \sqrt{2(a^2 + \max(a, 0) + 1)}$.
*2. Let $\mathbf{x} \in \mathbb{Z}^3$ with $x_3 \neq 0$. Then*
$\mathbf{x} \not\succeq (a, 0, 1) \implies \|\mathbf{x}\|_\infty < \sqrt{2(a^2 + \max(a, 0) + 1)}$.

3. *Let* $\mathbf{x} \in \mathbb{Z}^3$ *with* $x_2 \neq 0$ *and* $x_3 \neq 0$. *Then*
$\mathbf{x} \not\succeq (a, \operatorname{sgn}(x_2x_3), 1) \implies \|\mathbf{x}\|_\infty < \sqrt{2(a^2 + |a| + \max(a, 0) + 3)}$
*(where* sgn *designates the sign function).*

*Proof.* The conditions on $\mathbf{x}$ imply $f(\mathbf{x}) \geq f(\mathbf{x}_0)\ \forall f \in M_1$, where $\mathbf{x}_0 = (a, 1, 0)$, $(a, 0, 1)$, $(a, \operatorname{sgn}(x_2x_3), 1)$ in the cases 1,2,3 of the lemma respectively. For all $\mathbf{x}$ satisfying the assumptions we have

$$\begin{aligned} \mathbf{x} \not\succeq \mathbf{x}_0 &\iff \exists f \in V : & f(\mathbf{x}) < f(\mathbf{x}_0) \\ &\iff \exists f \in M_1 \cup M_2 : & f(\mathbf{x}) < f(\mathbf{x}_0) \\ &\iff \exists f \in M_2 : & f(\mathbf{x}) < f(\mathbf{x}_0) \end{aligned}$$

Now the assumption follows from Lemma 2.2 and estimating $f(\mathbf{x}_0)$ for $f \in M_2$. □

**Definition 2.5** *For* $a \in \mathbb{Z}$ *let*

$$\begin{aligned} \mathbb{Z}^3_* &= \left\{\mathbf{x} \in \mathbb{Z}^3 \setminus \{0\} : \ \gcd(\mathbf{x}) = 1 \text{ and } (\forall i > k : x_i = 0) \implies x_k \geq 0\right\} \\ W_{1,a} &= \left(\mathbb{Z}^3_*\right)^{\not\succeq\{(a,1,0)\}} \cup \{(a, 1, 0)\} \\ W_{2,a} &= \left(\mathbb{Z}^3_* \setminus \langle \mathbf{e}_1, \mathbf{e}_2\rangle_{\mathbb{Z}}\right)^{\not\succeq\{(a,0,1)\}} \cup \{(a, 0, 1)\} \\ W_{3,a} &= \left(\mathbb{Z}^3_* \setminus \left(\langle \mathbf{e}_1, \mathbf{e}_2\rangle_{\mathbb{Z}} \cup \langle \mathbf{e}_1, \mathbf{e}_3\rangle_{\mathbb{Z}}\right)\right)^{\not\succeq\{(a,-1,1),(a,1,1)\}} \cup \{(a, -1, 1), (a, 1, 1)\} \end{aligned}$$

Because of the symmetry $f(\mathbf{x}) = f(-\mathbf{x})$ and $f(k\mathbf{x}) = k^2 f(\mathbf{x})$ we can restrict ourselves to representation numbers with respect to $\mathbb{Z}^3_*$. The numbers $A_{\mathbb{Z}^3}(f, t)$ for $t \leq t_0$ determine the numbers $A_{\mathbb{Z}^3_*}(f, t)$ for $t \leq t_0$ and vice versa. Lemma 2.4 implies

$$\begin{aligned} W_{i,a} &\subset \left\{\mathbf{x} \in \mathbb{Z}^3_* : \|\mathbf{x}\|^2_\infty < 2\left(a^2 + \max(a, 0) + 1\right)\right\} \qquad \text{for } i = 1, 2 \\ W_{3,a} &\subset \left\{\mathbf{x} \in \mathbb{Z}^3_* : \|\mathbf{x}\|^2_\infty < 2\left(a^2 + |a| + \max(a, 0) + 3\right)\right\} \end{aligned}$$

**Lemma 2.6** *Let* $\emptyset \neq Y \subset X \subset \mathbb{Z}^3_*$ *and* $W \supset X^{\not\succeq Y} \cup Y$.
*Then* $\operatorname{MIN}(X) = \operatorname{MIN}(X \cap W)$.

The Lemma is a consequence of $\preceq$ being an order relation.

**Corollary 2.7** *For* $X \subset \mathbb{Z}^3_*$ *the following holds:*

1. $(a, 1, 0) \in X \implies \operatorname{MIN}(X) = \operatorname{MIN}(X \cap W_{1,a})$.
2. $X \subset \mathbb{Z}^3_* \setminus \langle \mathbf{e}_1, \mathbf{e}_2\rangle_{\mathbb{Z}}$ and $(a, 0, 1) \in X \implies \operatorname{MIN}(X) = \operatorname{MIN}(X \cap W_{2,a})$.
3. $X \subset \mathbb{Z}^3_* \setminus \left(\langle \mathbf{e}_1, \mathbf{e}_2\rangle_{\mathbb{Z}} \cup \langle \mathbf{e}_1, \mathbf{e}_3\rangle_{\mathbb{Z}}\right)$ and $\{(a, -1, 1), (a, 1, 1)\} \subset X$
$\implies \operatorname{MIN}(X) = \operatorname{MIN}(X \cap W_{3,a})$.

**Definition 2.8** *Let* $X \subset \mathbb{Z}^3_*$, *and* $\mathbf{x}_1, \ldots, \mathbf{x}_k \in X$ *pairwise different. We define* $K(X, \mathbf{x}_1 \ldots \mathbf{x}_k)$ *to be the set of forms in* $V$ *with (not necessarily linear independent) successively minimal vectors* $\mathbf{x}_1, \ldots, \mathbf{x}_k \in X$, *i.e.*

$$K(X, \mathbf{x}_1 \ldots \mathbf{x}_k) = \{f \in \overline{V} : f(\mathbf{x}_i) = \min f\left(X \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}\}\right) \quad \forall i = 1 \ldots k\}$$

*For* $k = 0$ *let* $K(X) = V$.

It is easy to see that

$$K(X, \mathbf{x}_1 \ldots \mathbf{x}_k) = \{f \in \overline{V} \ : \ f(\mathbf{x}_1) \le \ldots \le f(\mathbf{x}_k) \ \text{ and } \\ f(\mathbf{x}_k) \le f(\mathbf{x}) \quad \forall\, \mathbf{x} \in \mathrm{MIN}\big(X \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}\big)\} \tag{3}$$

We will always consider $K(X, \mathbf{x}_1 \ldots \mathbf{x}_k)$ where $X$ is one of $\mathbb{Z}^3_*$, $\mathbb{Z}^3_* \setminus \langle \mathbf{e}_1, \mathbf{e}_2 \rangle_{\mathbb{Z}}$ or $\mathbb{Z}^3_* \setminus \big(\langle \mathbf{e}_1, \mathbf{e}_2 \rangle_{\mathbb{Z}} \cup \langle \mathbf{e}_1, \mathbf{e}_3 \rangle_{\mathbb{Z}}\big)$. By Corollary 2.7 we know $\mathrm{MIN}\big(X \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}\big) = \mathrm{MIN}\big((X \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}) \cap W_{i,a}\big)$ for appropriate $a \in \mathbb{Z}$. Since $\mathrm{MIN}\big(X \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}\big)$ is finite and $V$ is defined by finitely many linear inequalities the set $K(X, \mathbf{x}_1 \ldots \mathbf{x}_k)$ is the intersection of finitely many open or closed halfspaces.

**Lemma 2.9** *Let $X$, $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be as in Definition 2.8 and $X \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \neq \emptyset$. Then*

$$K(X, \mathbf{x}_1 \ldots \mathbf{x}_k) = \bigcup_{\mathbf{y} \in \mathrm{MIN}(X \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_k\})} K(X, \mathbf{x}_1 \ldots \mathbf{x}_k, \mathbf{y})$$

*Proof.* Let $f \in K(X, \mathbf{x}_1 \ldots \mathbf{x}_k)$ and

$$Y_f = \{\mathbf{x} \in X \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \ : \ f(\mathbf{x}) = \min f\, \big(X \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}\big)\}$$

Then $\emptyset \neq \mathrm{MIN}(Y_f) \subset \mathrm{MIN}(X \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_k\})$. Now for any $\mathbf{y} \in \mathrm{MIN}(Y_f)$ we have $f \in K(X, \mathbf{x}_1 \ldots \mathbf{x}_k, \mathbf{y})$. The other inclusion is trivial. □

## 3 Coverings of $D$ by polyhedral cones

Consider the $n$-dimensional Euclidean space with scalar product ‘ · ’.

**Definition 3.1** *Let $T \subset \mathbb{R}^n$. We say that $T$ is a rational polyhedral cone iff there are finite sets $A, B \subset \mathbb{Q}^n$ such that*

$$T = T^n(A, B) = \bigcap_{\mathbf{a} \in A} \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{x} \cdot \mathbf{a} \ge 0\} \cap \bigcap_{\mathbf{b} \in B} \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{x} \cdot \mathbf{b} > 0\}$$

*The dimension of $T$ is the smallest dimension of a linear subspace containing $T$.*

Given $A, B$ and a rational subspace $U \subset \mathbb{R}^n$ there are well known algorithms to decide whether $T(A, B) \subset U$. In fact, we always compute the edges of $T$, i.e. a minimal finite set $E(T) = \{\mathbf{x}_1\mathbb{R}^+_0, \ldots, \mathbf{x}_r\mathbb{R}^+_0\}$ with $T = \sum_{i=1}^r \mathbf{x}_i\mathbb{R}^+_0$.

From now on we consider polyhedral cones $T \subset \mathbb{R}^6 \times \mathbb{R}^6$ of pairs of almost reduced forms, i.e. $T \subset \overline{V} \times \overline{V} \subset \mathbb{R}^{12}$. The intersection $T \cap (V \times V)$ is in general not a polyhedral cone, because the conditions (on both forms) $f \in V$ imply finitely many conditions $(U_\nu, \mathbf{b}_\nu)$ of the kind $f \in U_\nu \implies f \cdot \mathbf{b}_\nu > 0$ for some subspaces $U_\nu \subset \mathbb{R}^6$ and $\mathbf{b}_\nu \in \mathbb{R}^6$ with indices, say $\nu \in J$, given by the reduction conditions.

**Lemma 3.2** *Given $T = T^{12}(A, B) \subset \overline{V} \times \overline{V}$ we find a polyhedral cone $T_{V \times V}$ with*

$$T \cap (V \times V) \ \subset \ T_{V \times V} \ \subset \ T \qquad \textit{and} \qquad \overline{T \cap (V \times V)} = \overline{T_{V \times V}}$$

*Proof.* We construct the set $T_{V\times V}$ by taking into account only those reduction conditions that affect the closure. Let $I_1(T) = \{\nu \in J \;:\; T \subset U_\nu \times \mathbb{R}^6\}$ and $I_2(T) = \{\nu \in J \;:\; T \subset \mathbb{R}^6 \times U_\nu\}$. Let

$$\phi(T) = T \cap \bigcap_{\nu\in I_1} \{(f,g) \;:\; f \cdot \mathbf{b}_\nu > 0\} \cap \bigcap_{\nu\in I_2} \{(f,g) \;:\; g \cdot \mathbf{b}_\nu > 0\}$$

Clearly $T \supset \phi(T) \supset T \cap (V \times V)$. Consider the sequence $T_0 = T$, $T_{i+1} = \phi(T_i)$. This sequence becomes stable since $\dim \phi(T) = \dim T \implies \phi(\phi(T)) = \phi(T)$. The limit has the properties claimed for $T_{V\times V}$. □

Now we define a sequence of finite coverings $\mathfrak{T}_i$ of $D$ with polyhedral cones where $\mathfrak{T}_{i+1}$ is finer than $\mathfrak{T}_i$ and

$$\bigcap_{i\in\mathbb{N}_0} \bigcup_{T\in\mathfrak{T}_i} T \cap (V \times V) = D$$

Let $\mathfrak{T}_0 = \{(\overline{V} \times \overline{V})_{V\times V}\}$. Let $\mathfrak{T}$ be any covering of $D$ such that for each $T \in \mathfrak{T}$ there is a $k = k(T) \in \mathbb{N}_0$ with

P1: $T$ is a polyhedral cone and $T = T_{V\times V}$.

P2: Let $\Lambda = \Lambda(T)$ be one of $\emptyset$, $\langle \mathbf{e}_1, \mathbf{e}_2\rangle_{\mathbb{Z}}$, $\langle \mathbf{e}_1, \mathbf{e}_2\rangle_{\mathbb{Z}} \cup \langle \mathbf{e}_1, \mathbf{e}_3\rangle_{\mathbb{Z}}$ and let $\Lambda$ be maximal with $f_{|\Lambda} = g_{|\Lambda}\ \forall\, (f,g) \in T$. Let $X = \mathbb{Z}^3_* \setminus \Lambda$. There are $\mathbf{x}_1 \dots \mathbf{x}_k, \mathbf{y}_1 \dots \mathbf{y}_k \in X$ (belonging to $T$) with

$$\begin{aligned} T &\subset K(X, \mathbf{x}_1 \dots \mathbf{x}_k) \times K(X, \mathbf{y}_1 \dots \mathbf{y}_k) \\ T &\subset \{(f,g) \in \overline{V} \times \overline{V} \;:\; f(\mathbf{x}_\nu) = g(\mathbf{y}_\nu)\ \forall \nu = 1, \dots, k(T)\} \end{aligned}$$

Then a refinement $\mathfrak{T}'$ of $\mathfrak{T}$ may be constructed as follows:
Let $T \in \mathfrak{T}$ with $k, \Lambda, \mathbf{x}_1 \dots \mathbf{x}_k, \mathbf{y}_1 \dots \mathbf{y}_k$ as in P2. We define a covering $\mathfrak{M}_T$ of $T \cap D$ by:

- If $T \subset \Delta$ let $\mathfrak{M}_T = \{T\}$ (with the same $k$)
- If $T \not\subset \Delta$: For $\mathbf{x} \in \mathrm{MIN}(X \setminus \{\mathbf{x}_1, \dots, \mathbf{x}_k\})$, $\mathbf{y} \in \mathrm{MIN}(X \setminus \{\mathbf{y}_1, \dots, \mathbf{y}_k\})$ let

$$\begin{aligned} S_{\mathbf{xy}} &= [T \cap (K(X, \mathbf{x}_1 \dots \mathbf{x}_k, \mathbf{x}) \times K(X, \mathbf{y}_1 \dots \mathbf{y}_k, \mathbf{y}))]_{V\times V} \\ T_{\mathbf{xy}} &= [S_{\mathbf{xy}} \cap \{(f,g) \in \overline{V} \times \overline{V} \;:\; f(\mathbf{x}) = g(\mathbf{y})\}]_{V\times V} \end{aligned}$$

  Let $\Lambda_{\mathbf{xy}} = \Lambda(T_{\mathbf{xy}}) \in \{\emptyset, \langle \mathbf{e}_1, \mathbf{e}_2\rangle_{\mathbb{Z}}, \langle \mathbf{e}_1, \mathbf{e}_2\rangle_{\mathbb{Z}} \cup \langle \mathbf{e}_1, \mathbf{e}_3\rangle_{\mathbb{Z}}\}$ be maximal with $f_{|\Lambda_{\mathbf{xy}}} = g_{|\Lambda_{\mathbf{xy}}}\ \forall\, (f,g) \in T_{\mathbf{xy}}$. Let $X_{\mathbf{xy}} = \mathbb{Z}^3_* \setminus \Lambda_{\mathbf{xy}}$.
  - If $\Lambda_{\mathbf{xy}} = \Lambda$ let $k(T_{\mathbf{xy}}) = k(T) + 1$.
  - If $\Lambda_{\mathbf{xy}} \neq \Lambda$: Let $\mathbf{x}_{k+1} = \mathbf{x}$, $\mathbf{y}_{k+1} = \mathbf{y}$. Let $0 \leq r \leq k+1$ be maximal with $\sharp\left(\{\mathbf{x}_1, \dots, \mathbf{x}_r\} \setminus \Lambda_{\mathbf{xy}}\right) = \sharp\left(\{\mathbf{y}_1, \dots, \mathbf{y}_r\} \setminus \Lambda_{\mathbf{xy}}\right) = g$ and let $k(T_{\mathbf{xy}}) = g$.

  Define

$$\mathfrak{M}_T = \dot{\bigcup_{\substack{\mathbf{x}\in\mathrm{MIN}(X\setminus\{\mathbf{x}_1,\dots,\mathbf{x}_k\}) \\ \mathbf{y}\in\mathrm{MIN}(X\setminus\{\mathbf{y}_1,\dots,\mathbf{y}_k\})}}} \{T_{\mathbf{xy}}\} \qquad (4)$$

where $\dot{\bigcup}$ denotes the disjoint union. All $T_{\mathbf{xy}} \in \mathfrak{M}_T$ have properties P1 and P2 (with the number $k(T_{\mathbf{xy}})$ we defined) and we have

$$T \supset \bigcup_{\substack{\mathbf{x}\in \mathrm{MIN}(X\setminus\{\mathbf{x}_1,\dots,\mathbf{x}_k\}) \\ \mathbf{y}\in \mathrm{MIN}(X\setminus\{\mathbf{y}_1,\dots,\mathbf{y}_k\})}} \{S_{\mathbf{xy}}\} \supset T \cap (V \times V)$$

Let $(f, g) \in S_{\mathbf{xy}} \cap D$. Since $f_{|\Lambda} = g_{|\Lambda}$ we have $A_X(f, t) = A_X(g, t) \forall\, t \in \mathbb{R}_0^+$ and

$$f(\mathbf{x}) = \min f\left(X \setminus \{\mathbf{x}_1, \dots, \mathbf{x}_k\}\right) = \min\Big\{t_0 \in \mathbb{R}_0^+ : \sum_{0\le t\le t_0} A_X(f, t) \ge k + 1\Big\}$$
$$= g(\mathbf{y})$$

Thus

$$T \supset \bigcup_{\substack{\mathbf{x}\in \mathrm{MIN}(X\setminus\{\mathbf{x}_1,\dots,\mathbf{x}_k\}) \\ \mathbf{y}\in \mathrm{MIN}(X\setminus\{\mathbf{y}_1,\dots,\mathbf{y}_k\})}} \{T_{\mathbf{xy}}\} \supset T \cap D$$

Finally let

$$\mathfrak{T}' = \dot{\bigcup_{T\in\mathfrak{T}}} \mathfrak{M}_T \tag{5}$$

By construction $\mathfrak{T}'$ is a refinement of $\mathfrak{T}$ that covers $D$ and each $T \in \mathfrak{T}'$ has properties P1 and P2 with the number $k(T)$ we defined above.

**Definition 3.3** *Let $(\mathfrak{T}_i)$ be the series starting with $\mathfrak{T}_0 = \left\{(\overline{V} \times \overline{V})_{V\times V}\right\}$ as above and $\mathfrak{T}_{i+1} = \mathfrak{T}_i' \quad \forall i \ge 0$.*

The sets $T_{\mathbf{xy}}$ and $T'_{\mathbf{x}'\mathbf{y}'}$ coming from different $T$ or different pairs $(\mathbf{x}, \mathbf{y})$ may have the same elements but the construction may lead to different numbers $k(T_{\mathbf{xy}})$ and $k(T'_{\mathbf{x}'\mathbf{y}'})$. For notational convenience (to obtain a function $k(T)$) we defined $\mathfrak{M}_T$ and $\mathfrak{T}'$ (in (4),(5)) to be the disjoint union of the belonging $T_{\mathbf{xy}}$, thus identifying the members $T \in \mathfrak{T}_i$ not only by their elements but also by the way they are constructed.

**Definition 3.4** *Given $f \in V$, $k \in \mathbb{N}^{>0}$ let*

$$\psi(f, k) = \max\Big\{t \in f(\mathbb{Z}_*^3) \cup \{0\} : \sum_{s\le t} A_{\mathbb{Z}_*^3}(f, s) < k\Big\}$$

This is well defined since $f$ is positive definite and for each $f \in V$ we have $\psi(f, k) \overset{k\to\infty}{\longrightarrow} \infty$.

**Lemma 3.5** *Let $i \ge 3$, $\Delta \not\subset T \in \mathfrak{T}_i$, $(f, g) \in T \cap (V \times V)$. Then*

$$A_{\mathbb{Z}_*^3}(f, t) = A_{\mathbb{Z}_*^3}(g, t) \ \ \forall t \le \psi\left(f, [i/3]\right)$$

*where $[x]$ denotes the biggest integer $\le x$.*

*Proof.* Let $T_0 \supset \ldots \supset T_{i-1} \supset T_i = T$ be the sets $T_j \in \mathfrak{T}_j$ above $T$. Let $k_j = k(T_j)$, $\Lambda_j = \Lambda(T_j)$ as in P2. The sequence $\left(\Lambda_j\right)$ increases and has at most 3 values. Thus there is a section $\Lambda_\nu = \ldots = \Lambda_\mu$ of length $\mu - \nu + 1 \geq (i+1)/3$. Since $T_j \not\subset \Delta$ and $\Lambda_j = \Lambda_{j+1}$ implies $k_{j+1} = k_j + 1$ we have $k_\mu = \mu - \nu + k_\nu \geq \lceil i/3 \rceil$. Let $\mathbf{x}_1 \ldots \mathbf{x}_{k_\mu}, \mathbf{y}_1 \ldots \mathbf{y}_{k_\mu} \in \mathbb{Z}_*^3 \setminus \Lambda_\mu$ such that P2 holds. For all $t < f(\mathbf{x}_{k_\mu})$ holds

$$\begin{aligned} A_{\mathbb{Z}_*^3}(f,t) &= A_{\Lambda_\mu \cap \mathbb{Z}_*^3}(f,t) \quad + A_{\mathbb{Z}_*^3 \setminus \Lambda_\mu}(f,t) \\ &= \quad \text{"} \quad + \sharp\left\{1 \leq j < k_\mu \ : \ f(\mathbf{x}_j) = t\right\} \\ &= A_{\Lambda_\mu \cap \mathbb{Z}_*^3}(g,t) \quad + A_{\mathbb{Z}_*^3 \setminus \Lambda_\mu}(g,t) = A_{\mathbb{Z}_*^3}(g,t) \end{aligned}$$

Now the assertion follows from $\psi\left(f, \lceil i/3 \rceil\right) \leq \psi(f, k_\mu) < f(\mathbf{x}_{k_\mu})$. □

**Corollary 3.6** $\bigcap_{i \in \mathbb{N}_0} \bigcup_{T \in \mathfrak{T}_i} T \cap (V \times V) = D.$

## 4 Results

By explicitly computing the sequence $(\mathfrak{T}_i)$ we get the following

**Theorem 4.1** *The sequence* $(\mathfrak{T}_i)$ *becomes stable and for* $i \geq 14$ *we have* $T \subset \Delta \ \ \forall\, T \in \mathfrak{T}_i$. *Thus* $D \subset \Delta$.

We can extract from the computation a bound $b(f)$ such that for all $f, g \in V$ holds $A_{\mathbb{Z}_*^3}(f,t) = A_{\mathbb{Z}_*^3}(g,t) \ \ \forall t \leq b(f) \implies f = g$.

**Definition 4.2** *For a positive definite n-dimensional quadratic form f let*

$$s_i(f) = \min\left\{f(\mathbf{x}) \ : \ \exists \mathbf{x}_1, \ldots, \mathbf{x}_i \in \mathbb{Z}^3 \text{ lin. independent with } f(\mathbf{x}_\nu) \leq f(\mathbf{x})\right\}$$

*be the successive minima of f.*

For a Minkowski reduced form $f$ of dimension $\leq 4$ we have $s_i(f) = f_{ii}$ (see e.g.[Wae]). We shall find an optimal linear bound in the diagonal coefficients of the kind

$$b(f) = \min\left\{\sum_{\nu=1}^{3} r_\nu f_{\nu\nu} \ : \ (r_1, r_2, r_3) \in R \subset \mathbb{Q}^3\right\}$$

where $R$ is a finite set. This is done by collecting all conditions $f(\mathbf{x}) = g(\mathbf{y})$ we need during the process of refining $\mathfrak{T}_i$ to $\mathfrak{T}_{i+1}$ up to $i = 14$. More precisely: With $\Delta \not\supset T \in \mathfrak{T}_i$, $k, \mathbf{x}_1 \ldots \mathbf{x}_k, \mathbf{y}_1 \ldots \mathbf{y}_k$ belonging to $T$ (satisfying P2) and $\mathbf{x} \in \mathrm{MIN}\left(X \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}\right)$ let

$$S_{\mathbf{x}} = \left[T \cap \left(K(X, \mathbf{x}_1 \ldots \mathbf{x}_k, \mathbf{x}) \times \overline{V}\right)\right]_{V \times V}$$

$$C(T) = \bigcap_{\substack{\mathbf{x} \in \mathrm{MIN}\left(X \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}\right) \\ S_{\mathbf{x}} \not\subset \Delta}} \left\{(r_1, r_2, r_3) \in \mathbb{Q}^3 \ : \ \sum_{\nu=1}^{3} r_\nu f_{\nu\nu} \geq f(\mathbf{x}) \ \forall\, (f,g) \in S_{\mathbf{x}}\right\}$$

In the last definition we can replace $(f,g) \in S_{\mathbf{x}}$ by $(f,g) \in E(S_{\mathbf{x}})$ (the set of edges of the polyhedral cone $S_{\mathbf{x}}$). Define a series $(C_i)_{i \in \mathbb{N}_0}$ by $C_0 = \mathbb{Q}^3$ and

$$C_i = C_{i-1} \cap \bigcap_{\substack{T \in \mathfrak{T}_{i-1} \\ T \not\subset \Delta}} C(T)$$

For a set $C \subset \mathbb{Q}^3$ let

$$D_C = \left\{(f,g) \in V \times V \ : \ A_{\mathbb{Z}^3_*}(f,t) = A_{\mathbb{Z}^3_*}(g,t) \ \ \forall t \le \inf_{r \in C} \sum_{\nu=1}^{3} r_\nu f_{\nu\nu} \right\}$$

Clearly for $\mathbb{Q}^3 \supset C \supset C'$ we have $D_C \supset D_{C'} \supset D$. By an inductive argument we see

**Lemma 4.3** *For all $i \le j \in \mathbb{N}_0$ the following holds:* $\bigcup_{T \in \mathfrak{T}_i} T \ \supset \ D_{C_j}$.

Again the sets $C_i$ can be explicitly computed since they are determined by finite sets of linear inequalities with rational coefficients. For $i_0 = 14$ the set $C_{i_0}$ is bounded and its elements are convex combinations of a finite set of vertices $V(C_{i_0})$. This yields

**Theorem 4.4** *Let $f, g$ be ternary positive definite forms with real coefficients and let $s_i = s_i(f)$ be the successive minima of $f$. Let*

$$\begin{aligned} b(f) = \min\{ & -1/14\, s_1 + 18/7\, s_2 + s_3 \quad , \quad 3/2\, s_1 - 5/6\, s_2 + 17/6\, s_3, \\ & 13/5\, s_1 + s_2 + s_3 \quad , \quad 7/2\, s_3 \} \end{aligned}$$

*and* $\quad A(f,t) = A(g,t) \ \ \forall t \le b(f)$.
*Then $f$ and $g$ are integrally equivalent.*

*Remark.* The first three terms in the definition of $b(f)$ come from the three vertices of $C_{14}$, the term $7/2\, s_3$ is redundant (but the optimal bound only involving $s_3$).

Now all computations can be redone introducing the condition $\det f = \det g$ i.e. we define $\mathfrak{M}_T = \{T\}$ iff $T \cap \{(f,g) \ : \ \det f = \det g\} \cap (V \times V) \subset \Delta$ and in the other case we replace (4) by $\mathfrak{M}_T = \dot{\bigcup}_{T_{\mathbf{xy}} \cap \{(f,g) :\, \det f = \det g\} \neq \emptyset} \{T_{\mathbf{xy}}\}$. The conditions involving the determinants were checked by a heuristic decision procedure that works on all sets we are dealing with. Making the appropriate changes to the definition of $C_i$ and computing the coverings $(\mathfrak{T}_i)$ and $V(C_i)$ with the determinant conditions we get:

**Theorem 4.5** *Let $f, g, s_i$ be as in Theorem* 4.4 *with* $\det f = \det g$. *Let*

$$\begin{aligned} b(f) = \min\{ & \quad s_1 - s_2 + 3\, s_3 \quad , \quad 11/13\, s_1 - 6/13\, s_2 + 34/13\, s_3, \\ & -s_1 + 2\, s_2 + 2\, s_3 \quad , \quad 4/3\, s_1 + 1/3\, s_2 + 5/3\, s_3, \\ & -2/3\, s_1 + 3\, s_2 + s_3 \quad , \quad 14/9\, s_1 + s_2 + s_3 \quad , \quad 3\, s_3 \} \end{aligned}$$

*and* $\quad A(f,t) = A(g,t) \ \ \forall t \le b(f)$.
*Then $f$ and $g$ are integrally equivalent.*

*Remark about the computations*

Determining the sets MIN($X$) we are always in the situation of Corollary 2.7. We precompute the sets $W_{i,a}$ for $a = 3$ which turns out to be sufficient. The vectors from these sets are ordered in a tree structure that reflects the relation $\preceq$ and makes the computations of the involved MIN($X$) much faster. The actual implementation that determines $\bigcup_{T \in \mathfrak{T}_i} T$ uses a more elaborate partition of each $T \cap D$ (with $T \in \mathfrak{T}_i$) into disjoint sets (introducing additional strict inequalities on the boundary of the subsets). It also takes advantage of the symmetry induced by exchanging $f$ and $g$ in the pairs $(f, g) \in \overline{V} \times \overline{V}$. For a description of the algorithms see [Sch2]. To determine $\bigcup_{T \in \mathfrak{T}_{14}} T$ we computed about 120000 polyhedral cones of different dimensions (most of them of dimension 1 or 2). The analogous computation with the additional condition on the determinants involves about 30000 polyhedral cones. Both jobs needed 5-10 hours CPU time.

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