THE THIRTY-NINE SYSTEMS OF QUATERNIONS WITH A POSITIVE NORM-FORM AND SATISFACTORY FACTORABILITY

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1. Introduction. The quaternion arithmetics which we shall set forth are of particular interest because for them, and only for them, among systems of rational generalized quaternions with positive definite norm-forms, is factorization always possible and unique, under conditions rather like those for integral Hamiltonian quaternions [5]. It may be surmised therefore that these systems will be susceptible of many applications, and it is our purpose here to tabulate the facts about the individual 39 systems in order to facilitate their use. The systems themselves, and the proof of their unique properties in §5, were derived by Pall [5].

2. Definitions and notations. Quaternions (generalized) [4; §3] are quantities of the form $t = t_0 + i_1t_1 + i_2t_2 + i_3t_3$ where the coordinates t_i range over some field, say that of reals, and the basal elements $1, i_1, i_2, i_3$ satisfy a multiplication table associated as follows with a given symmetric ternary matrix $(a_{\alpha\beta})$ and the adjoint matrix $(A_{\alpha\beta})$:

$$i_\alpha^2 = -A_{\alpha\alpha} \ (\alpha = 1, 2, 3); \qquad i_\alpha i_\beta = -A_{\alpha\beta} + \sum_{\delta=1}^{3} a_{\gamma\delta} i_\delta ,$$

$$i_\beta i_\alpha = -A_{\beta\alpha} - \sum_{\delta} a_{\gamma\delta} i_\delta ,$$

α, β, γ being a cyclic permutation of 1, 2, 3. The fundamental number of the system is $d = 4 \mid a_{\alpha\beta} \mid$ and is assumed not zero. The case $(a_{\alpha\beta}) = I$, the identity matrix, gives the Hamiltonian quaternions.

A suitable basis for integral elements [5; §3] is given, in case the $a_{\alpha\alpha}$ and $2a_{\alpha\beta}$ are rational integers, by the quantities $1, j_1, j_2, j_3$, where

$$j_\alpha = i_\alpha + \tfrac{1}{2}\epsilon_\alpha \qquad (\alpha = 1, 2, 3);$$

and $\epsilon_\alpha = 0$ if $2a_{\beta\gamma}$ is even, $\epsilon_\alpha = 1$ if $2a_{\beta\gamma}$ is odd. Thus $t = t_0 + \sum i_\alpha t_\alpha = t_0' + \sum j_\alpha t_\alpha$ (whence $t_0' = t_0 - \frac{1}{2}\sum \epsilon_\alpha t_\alpha$) is integral if and only if t_1, t_2, t_3, and $t_0 - \frac{1}{2}\sum \epsilon_\alpha t_\alpha$ are integers.

It is assumed in this article that the form $f = \sum a_{\alpha\beta}x_\alpha x_\beta$ has integral coefficients $a_{\alpha\alpha}$ and $2a_{\alpha\beta}$, and is positive definite; $\mathfrak{F}$ denotes the adjoint form, of matrix $(A_{\alpha\beta})$, whose elements are in general not integers. However, the quaternary form $F(t_0', t_1, t_2, t_3) = (t_0' + \frac{1}{2}\sum \epsilon_\alpha t_\alpha)^2 + \sum A_{\alpha\beta}t_\alpha t_\beta$ has integral co-

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efficients. The last form gives the norm of t; that is, if the conjugate $\bar{t}$ of t is defined by $t_0 - \sum i_\alpha t_\alpha$, then $\bar{t}t = t\bar{t} = F(t_0', t_1, t_2, t_3)$. It can be shown that the system of integral quaternions associated with $(a_{\alpha\beta})$ is maximal [5; §3] if and only if the form $\mathfrak{F}$ cannot be derived by an integral, linear transformation from the adjoint of an integral form of smaller determinant than f; and that this holds if and only if d is a squarefree integer and c_p is -1 for each prime in d. Only five of the 39 systems are maximal [5; Theorem 10].

3. **An example out of Table I.** A system with $d = 4 \mid a_{\alpha\beta} \mid$ is denoted by F_d, and several systems of equal determinant are distinguished by writing F_d, F_d', F_d'', etc. Table I gives for all 39 systems the ternary form f, the associated adjoint form, and the ϵ_α. This makes it easy to construct the multiplication table for any system, and to express the condition on the i-coordinates that a quaternion be integral.

For instance, if we consider F_{12}'':

$$(a_{\alpha\beta}) = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 2 \end{bmatrix}, \qquad (A_{\alpha\beta}) = \begin{bmatrix} 15/4 & -\frac{3}{4} & -\frac{3}{4} \\ -\frac{3}{4} & 7/4 & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{1}{4} & 7/4 \end{bmatrix}.$$

The multiplication table is $i_1^2 = -15/4$, $i_2^2 = i_3^2 = -7/4$,

$$i_1 i_2 = \tfrac{3}{4} + \tfrac{1}{2}i_1 + \tfrac{1}{2}i_2 + 2i_3, \qquad i_2 i_1 = \tfrac{3}{4} - \tfrac{1}{2}i_1 - \tfrac{1}{2}i_2 - 2i_3,$$

$$i_3 i_1 = \tfrac{3}{4} + \tfrac{1}{2}i_1 + 2i_2 + \tfrac{1}{2}i_3, \qquad i_2 i_3 = \tfrac{1}{4} + i_1 + \tfrac{1}{2}i_2 + \tfrac{1}{2}i_3, \text{ etc.}$$

Since $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, $t = t_0 + \sum i_\alpha t_\alpha = (t_0 - \frac{1}{2}\sum t_\alpha) + \sum j_\alpha t_\alpha$. Accordingly, t is integral if t_1, t_2, t_3 and $t_0 - \frac{1}{2}t_1 - \frac{1}{2}t_2 - \frac{1}{2}t_3$ are integers. The norm of t is $t_0^2 + \sum A_{\alpha\beta}t_\alpha t_\beta$, or

$$\begin{aligned} (1) \qquad & (t_0' + \tfrac{1}{2}t_1 + \tfrac{1}{2}t_2 + \tfrac{1}{2}t_3)^2 + \tfrac{1}{4}(15t_1^2 + 7t_2^2 + 7t_3^2 - 6t_1t_2 - 6t_1t_3 - 2t_2t_3) \\ & = t_0'^2 + t_0't_1 + t_0't_2 + t_0't_3 + 4t_1^2 + 2t_2^2 + 2t_3^2 - t_1t_2 - t_1t_3. \end{aligned}$$

Since the minimum of $\sum A_{\alpha\beta}t_\alpha t_\beta$ is $7/4$, the norm of t is 1 only if $t_1 = t_2 = t_3 = 0$, $t_0' = \pm 1$; hence ± 1 are the only units.

4. **Equivalent systems with simpler multiplication tables.** To obtain a simpler multiplication table we can apply a rational transformation replacing $(a_{\alpha\beta})$ by a diagonal matrix. In Table I the values of the c_p (the rational invariants of Hasse) [2] or [3; §4] are given for each form and enable us to determine to which diagonal forms $\mathfrak{F}$ is rationally equivalent. The system of integral quaternions in the original system transforms into a system of quaternions in the new system (also closed under addition and multiplication, etc.) and the conditions of integrality transform into certain conditions on the coefficients in the new

system. Table II lists for each of the 39 systems an arithmetically equivalent system with a diagonal multiplication table. The integrality conditions and the units are stated for these diagonal systems. There are, to be sure, infinitely many such equivalent systems, but we have tried to choose those with reasonably simple integrality conditions.

To illustrate the derivation of such a diagonal system, we again consider $F_{12}'' : c_p(\mathfrak{F}) = c_p(f) = (-3, -7)_p = -1$ only for $p = 3, \infty$. Hence $\mathfrak{F}$ is rationally equivalent to $3x_1^2 + 3x_2^2 + x_3^2$, since $(-3, -1)_p = -1$ only for $p = 3$ and ∞. Indeed, by (1) of §3,

$$F = (t_0' + \tfrac{1}{2}t_1 + \tfrac{1}{2}t_2 + \tfrac{1}{2}t_3)^2 + \tfrac{3}{4}(2t_1 - t_3)^2 + \tfrac{3}{4}(t_2 - t_1 - t_3)^2 + \tfrac{1}{4}(2t_2 + t_3)^2.$$

Hence if we set

$$y_0 = 2t_0' + t_1 + t_2 + t_3, \qquad y_1 = 2t_1 - t_3,$$

$$y_2 = -t_1 + t_2 - t_3, \qquad y_3 = 2t_2 + t_3,$$

then the norm-form becomes $(\tfrac{1}{2}y_0)^2 + 3(\tfrac{1}{2}y_1)^2 + 3(\tfrac{1}{2}y_2)^2 + (\tfrac{1}{2}y_3)^2$, and the system of quaternions $y = \tfrac{1}{2}(y_0 + i_1y_1 + i_2y_2 + i_3y_3)$, with the multiplication table

	i_1	i_2	i_3
i_1	-3	$3i_3$	$-i_2$
i_2	$-3i_3$	-3	i_1
i_3	i_2	$-i_1$	-1

given by $f = x_1^2 + x_2^2 + 3x_3^2$, is rationally equivalent to F_{12}''.

Furthermore, in the original system t is integral if t_0', t_1, t_2, t_3 are integers. So in the new system, y is integral if y_0, y_1, y_2, y_3 are integers such that $y_0 \equiv y_2$, mod 2, and $y_1 + 2y_2 + 3y_3 \equiv 0$, mod 8.

If we use the identities

$$\begin{aligned}15t_1^2 + 7t_2^2 + 7t_3^2 - 6t_1t_2 - 6t_1t_3 - 2t_2t_3 &\\ &= 3(t_1 + t_2 - t_3)^2 + 3(t_1 + t_3)^2 + (3t_1 - 2t_2 - t_3)^2\\ &= 3(t_1 - t_2 - t_3)^2 + 3(2t_1)^2 + (2t_2 - 2t_3)^2,\end{aligned}$$

we get two other equivalent systems with the same multiplication table, with quaternions of the form $\tfrac{1}{2}(y_0 + i_1y_1 + i_2y_2 + i_3y_3)$, and the respective integrality conditions (i) the y_i integral, $y_0 \equiv y_1$, mod 2, $2y_1 + 3y_2 - y_3 \equiv 0$, mod 8, (ii) the y_i integral, y_2 and y_3 even, $y_0 \equiv y_1$, mod 2, and $y_2 + y_3 \equiv 2y_1$, mod 4. All three systems thus found are arithmetically equivalent to F_{12}''.

5. Statement of theorems on factorability of integral quaternions. If $x = ut$ in integral quaternions, t is called a right-divisor of x. Integral quaternions of norm 1 (units) will be designated by the letter θ. The left-associates θt of t are, with t, right-divisors of x. A necessary condition is that Nt shall divide Nx. (Here Nt denotes the norm $t\bar{t}$ of t). We call x primitive if its j-coordinates are relatively prime. The following theorems refer to any system associated with an integral ternary f of non-zero determinant $\frac{1}{2}d$.

THEOREM 1. (*Uniqueness of factorability for all primitive quaternions*). *Let x be primitive. If $Nt = m$, and t is a right-divisor of x, the only right-divisors of x of norm m are the quaternions θt, provided*

(1) *m has no prime factor p such that $p^2 \mid d$ or such that $p \| d$ and $c_p = 1$.*

THEOREM 2. (*Existence of factors when the generally necessary conditions are satisfied.*) *Let m be a non-zero integer satisfying* (1) *and represented by F. Let x be primitive and $m \mid Nx$. Then if F is in a genus of one class, x has a unique set θt of right-divisors of norm m.*

THEOREM 3. *If F is not in a genus of one class, then there exist infinitely many primes p represented by F, and for each p primitive quaternions x of norms divisible by p but having no right-divisors of norm p.*

It should be noted that restriction (1) is vacuous [5; §3] in the maximal cases, and is otherwise a mild and generally necessary restriction. The 39 systems listed in Table I are precisely the cases of positive definite norm-forms in genera of one class.

We do not wish in this article to enter into many applications. However, we will prove as a consequence of Theorems 1 and 2 the existence of a greatest common right-divisor.

6. Existence of greatest common right-divisor.

THEOREM 4. *Let x be a primitive integral quaternion, y an integral quaternion. If F is in a genus of one class, and the prime factors p of Nx are represented by F and do not satisfy $p^2 \mid d$ or $p \mid d$ and $c_p = 1$, then there exists a greatest common right-divisor of x and y.*

Proof. We can evidently begin by writing

(1) $$x = ut, \qquad y = vt,$$

where u and v have no common right-divisors, (except units). We will now prove that t is a g.c.r.d. of x and y. If

(2) $$x = u't', \qquad y = v't',$$

we must prove that t' is a right-divisor of t. Let $m = Nt$, $m' = Nt'$. The desired result is obvious from the uniqueness of the right-divisor of norm m' of the primitive x, in case $m' \mid m$. Assume if possible the contrary, and write $m = dm''$, $m' = d'm''$, where $(d, d') = 1$. Then as the right-divisors of x with norm m'' are unique, we can set

$$(3) \qquad t = at'', \qquad t' = a't'', \qquad Na = d, \qquad Na' = d', \qquad Nt'' = m''.$$

Hence by (1) and (2), $ua = u'a'$, $va = v'a'$, whence

$$(4) \qquad Na \cdot u = u'a'\bar{a}, \qquad Na \cdot v = v'a'\bar{a}.$$

Hence the right-divisor of norm d' of the primitive quaternion $a'\bar{a}$ is a common right-divisor of $Na\ u$ and $Na\ v$, hence since Na is prime to d', a common right-divisor of u and v. It follows that $d' = 1$ and $m' \mid m$. (To see that $a'\bar{a}$ is primitive, note that $a'd$ is primitive, mod d', and $d'\bar{a}$ is primitive, mod d.)

7. **Genealogies of the 39 systems.** It is shown in reference [5] that only five of the thirty-nine systems are fundamental (corresponding to maximal arithmetics), namely F_2, F_3, F_5, F_7, and F_{13}. Indeed, all the systems (21 in number) with $c_2 = -1$ can be obtained from F_2 by an integral transformation, and similarly those with $c_3 = -1$ may be obtained from F_3 by an integral transformation, etc. However, there are further connections between the forms thus obtained. For instance, there is an integral transformation of determinant 2 that takes F_4 into F_8, but no such transformation taking F_4 into F_8'; it is necessary to apply a transformation of determinant 2^2 on F_2 to get F_8. It can be shown that it is never necessary to use a transformation of higher determinant than p^2 to derive one form from another [5; §12]. The genealogy, which appears below, shows completely the relations between the different forms. An arrow connecting one form to another means that the second may be derived from the first by an integral transformation.

To show how this genealogy was constructed we take a few examples. In certain cases it is easy to find a transformation which will take one form into another. Consider F_4, F_8.

$$F_4 = x_0^2 + x_1^2 + x_2^2 + x_3^2,$$

$$F_8 = y_0^2 + 2y_1^2 + 2y_2^2 + y_3^2 = y_0^2 + (y_1 - y_2)^2 + (y_1 + y_2)^2 + y_3^2.$$

Therefore, the transformation: $x_0 = y_0$, $x_1 = y_1 - y_2$, $x_2 = y_1 + y_2$, $x_3 = y_3$ takes F_4 into F_8. The equivalent diagonal systems criteria are useful in some cases. Let us take F_9 and F_{27}. In the diagonal equivalent of F_9, an integral quaternion is of the form $\frac{1}{2}(y_0 + i_1y_1 + i_2y_2 + 3i_3y_3)$, where the y_i are integers, and $y_0 \equiv y_2$, $y_1 \equiv y_3$, mod 2. In the diagonal equivalent of F_{27} an integral quaternion is of the form $\frac{1}{2}(y_0 + i_1y_1 + i_2y_2 + 3i_3y_3)$, where the y_i are integers and $y_1 \equiv -y_2$, mod 3, and $y_0 \equiv y_2$, $y_1 \equiv y_3$, mod 2. Therefore, the integral quaternions of F_{27} are just a subset of those belonging to F_9, and so F_{27} may be derived from F_9 by an integral transformation.

We will now show a couple of ways of proving that one form cannot be derived from another by an integral transformation. In some cases it is only necessary to consider the numbers represented by each form. F_8' does not represent any number of the form $4n + 2$, whereas F_{16}' does represent such numbers; therefore, there is no integral transformation taking F_8' into F_{16}'. Next consider F_{16}'', F_{32}'. F_{16}'' has the form-residue [5; §13] $x_0^2 + 3x_3^2 + 8(x_1^2 + x_1x_2 + x_2^2)$, mod 2^r; F_{32}' has the form-residue $(1, -3, -8, 24)$, mod 2^r. If there is a transformation T taking F_{16}'' into F_{32}' we may assume [5; §12]:

$$T = \begin{pmatrix} 2^\rho & \lambda & \mu \\ 0 & 2^\sigma & \nu \\ 0 & 0 & 2^\tau \end{pmatrix},$$

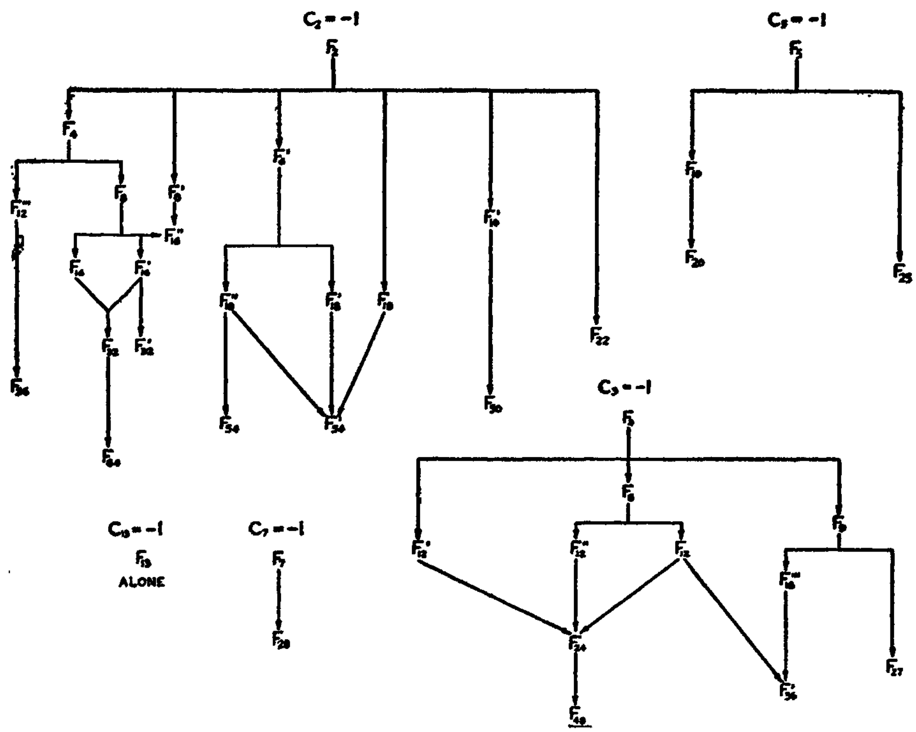
where $0 \leq \lambda, \mu \leq 2^\rho$, $0 \leq \nu < 2^\sigma$, $\rho + \sigma + \tau = 1$, $\rho, \sigma, \tau \geq 0$. Therefore, the only possibilities are:

(1) $x_1 \to 2x_1$, $\quad x_2 \to x_2$, $\quad x_3 \to x_3$.
(1′) $x_1 \to 2x_1 + x_2$, $\quad x_2 \to x_2$, $\quad x_3 \to x_3$.
(2) $x_1 \to 2x_1 + x_3$, $\quad x_2 \to x_2$, $\quad x_3 \to x_3$.
(3) $x_1 \to 2x_1 + x_2 + x_3$, $\quad x_2 \to x_2$, $\quad x_3 \to x_3$.
(4) $x_1 \to x_1$, $\quad x_2 \to 2x_2$, $\quad x_3 \to x_3$.
(5) $x_1 \to x_1$, $\quad x_2 \to 2x_2 + x_3$, $\quad x_3 \to x_3$.
(6) $x_1 \to x_1$, $\quad x_2 \to x_2$, $\quad x_3 \to 2x_3$.

(1), (4), (6) obviously do not take F_{16}'' into a form F_{32}', mod 2^r. (2) takes F_{16}'' into $x_0^2 + 3x_3^2 + 8(4x_1^2 + x_3^2 + 4x_1x_3 + 2x_1x_2 + x_2x_3 + x_2^2)$. This represents 11, and is therefore not $(1, -3, -8, 24)$, mod 2^r. For the same reason (3) does not work. (5) takes F_{16}'' into $x_0^2 + 3x_3^2 + 8(x_1^2 + 2x_1x_2 + x_1x_3 + 4x_2^2 + 4x_2x_3 + x_3^2)$ which is not $(1, -3, -8, 24)$, mod 2^r. Therefore, there is no integral transformation that takes F_{16}'' into F_{32}'.

It should perhaps be pointed out that Table II gives examples of interesting factorization systems. For example, consider F_{64}. It is well known that the system (corresponding to F_4) of ordinary quaternions $y = y_0 + i_1y_1 + i_2y_2 + i_3y_3$ where the y_i are integers, are not only closed under addition and multiplication, but also completely factorable in the sense that if $Ny = mn$, where m is odd, then y can be factored within the system as zw, with $Nz = m$ and $Nw = n$. But the example F_{64} shows that the same property holds true of the subsystem in which the coordinates y_i are even and have a sum divisible by 4.

GENEALOGIES OF THE 39 FORMS



Notations used in the tables. (a, b, c, r, s, t) denotes the ternary form of matrix

$$\begin{pmatrix} a & t & s \\ t & b & r \\ s & r & c \end{pmatrix}.$$

$\ni$ means "such that". In the first column of Table II (a, b, c) denotes $(a, b, c, 0, 0, 0)$; in the second column of Table II (a, b, c, d) denotes $ay_0 + by_1i_1 + cy_2i_2 + dy_3i_3$.

Table I

	Units (in j-coordinates)	All $p \ni p^2 \mid d$ or $p \parallel d$, $c_p = 1$
$F_2 : f = (1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$\pm 1, \pm j_\alpha, \pm(1 - j_\alpha)$,	
$\mathfrak{F} = \frac{1}{4}(3, 3, 3, -1, -1, -1)$	$\pm(1 - j_2 - j_3)$,	
	$\pm(1 - j_1 - j_3)$,	
$c_2 = -1. \quad \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$	$\pm(1 - j_1 - j_2)$,	
	$\pm(1 - j_1 - j_2 - j_3)$,	
	$\pm(2 - j_1 - j_2 - j_3)$.	

Table I—*Continued.*

	Units (in j-coordinates)	All $p \ni p^2 \mid d$ or $p \parallel d$, $c_p = 1$
$F_3: f = (1, 1, 1, -\frac{1}{2}, 0, 0)$	$\pm 1, \pm j_1, \pm j_3, \pm j_2,$	
$\mathfrak{F} = (\frac{3}{4}, 1, 1, \frac{1}{2}, 0, 0)$	$\pm(j_3 - j_2), \pm(1 - j_1).$	
$c_3 = -1.\ \epsilon_1 = 1, \epsilon_2 = \epsilon_3 = 0$		
$F_4: f = (1, 1, 1, 0, 0, 0)$	$\pm 1, \pm j_1, \pm j_2, \pm j_3.$	
$\mathfrak{F} = (1, 1, 1, 0, 0, 0)$		2
$c_2 = -1.\ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$		
$F_5: f = (1, 1, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$\pm 1, \pm(1 - j_3), \pm j_3.$	
$\mathfrak{F} = \frac{1}{4}(7, 7, 3, -1, -1, -3)$		
$c_5 = -1.\ \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$		
$F_6: f = (1, 1, 2, -\frac{1}{2}, -\frac{1}{2}, 0)$	$\pm 1, \pm j_3.$	
$\mathfrak{F} = \frac{1}{4}(7, 7, 4, 2, 2, 1)$		2
$c_3 = -1.\ \epsilon_1 = \epsilon_2 = 1, \epsilon_3 = 0$		
$F_6': f = (1, 1, 2, 0, 0, -\frac{1}{2})$	$\pm 1, \pm j_3, \pm(1 - j_3).$	
$\mathfrak{F} = (2, 2, \frac{3}{4}, 0, 0, 1)$		3
$c_2 = -1.\ \epsilon_1 = \epsilon_2 = 0, \epsilon_3 = 1$		
$F_7: f = (1, 1, 2, -\frac{1}{2}, 0, 0)$	$\pm 1, \pm j_3.$	
$\mathfrak{F} = (7/4, 2, 1, \frac{1}{2}, 0, 0)$		
$c_7 = -1.\ \epsilon_1 = 1, \epsilon_2 = \epsilon_3 = 0$		
$F_8: f = (1, 1, 2, 0, 0, 0)$	$\pm 1, \pm j_3.$	
$\mathfrak{F} = (2, 2, 1, 0, 0, 0)$		2
$c_2 = -1.\ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$		
$F_8': f = (1, 1, 3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$\pm 1, \pm j_3, \pm(1 - j_3).$	2
$\mathfrak{F} = \frac{1}{4}(11, 11, 3, -1, -1, -5)$		
$c_2 = -1.\ \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$		
$F_9: f = (1, 1, 3, 0, 0, -\frac{1}{2})$	$\pm 1, \pm j_3, \pm(1 - j_3).$	3
$\mathfrak{F} = (3, 3, \frac{3}{4}, 0, 0, \frac{3}{2})$		
$c_3 = -1.\ \epsilon_1 = \epsilon_2 = 0, \epsilon_3 = 1$		
$F_{10}: f = (1, 2, 2, 1, \frac{1}{2}, \frac{1}{2})$	$\pm 1.$	2
$\mathfrak{F} = (3, 7/4, 7/4, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{2})$		
$c_5 = -1.\ \epsilon_1 = 0, \epsilon_2 = \epsilon_3 = 1$		
$F_{10}': f = (1, 1, 3, -\frac{1}{2}, -\frac{1}{2}, 0)$	$\pm 1, \pm j_3.$	5
$\mathfrak{F} = (11/4, 11/4, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4})$		
$c_2 = -1.\ \epsilon_1 = \epsilon_2 = 1, \epsilon_3 = 0$		
$F_{12}: f = (1, 1, 3, 0, 0, 0)$	$\pm 1, \pm j_3.$	2
$\mathfrak{F} = (3, 3, 1, 0, 0, 0)$		
$c_3 = -1.\ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$		
$F_{12}': f = (1, 1, 4, 0, 0, -\frac{1}{2})$	$\pm 1, \pm j_3, \pm(1 - j_3).$	2
$\mathfrak{F} = (4, 4, \frac{3}{4}, 0, 0, 2)$		
$c_3 = -1.\ \epsilon_1 = \epsilon_2 = 0, \epsilon_3 = 1$		

Table I—*Continued.*

	Units (in j-coordinates)	All $p \ni p^2 \mid d$ or $p \parallel d$, $c_p = 1$
$F_{12}'' : f = (1, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	± 1.	2
$\mathfrak{F} = \frac{1}{4}(15, 7, 7, -1, -3, -3)$		
$c_3 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$		
$F_{12}''' : f = (1, 2, 2, -1, 0, 0)$	± 1.	2, 3
$\mathfrak{F} = (3, 2, 2, 1, 0, 0)$		
$c_2 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$		
$F_{13} : f = (1, 2, 2, -\frac{1}{2}, 0, -\frac{1}{2})$	± 1.	
$\mathfrak{F} = (15/4, 2, 7/4, \frac{1}{2}, \frac{1}{4}, 1)$		
$c_{13} = -1.\ \ \epsilon_1 = 1, \epsilon_2 = 0, \epsilon_3 = 1$		
$F_{16} : f = (1, 2, 2, 0, 0, 0)$	± 1.	2
$\mathfrak{F} = (4, 2, 2, 0, 0, 0)$		
$c_2 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$		
$F_{16}' : f = (1, 1, 4, 0, 0, 0)$	$\pm 1, \pm j_3$.	2
$\mathfrak{F} = (4, 4, 1, 0, 0, 0)$		
$c_2 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$		
$F_{16}'' : f = (2, 2, 2, 1, 1, 1)$	± 1.	2
$\mathfrak{F} = (3, 3, 3, -1, -1, -1)$		
$c_2 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$		
$F_{18} : f = (1, 1, 5, -\frac{1}{2}, -\frac{1}{2}, 0)$	$\pm 1, \pm j_3$.	3
$\mathfrak{F} = \frac{1}{4}(19, 19, 4, 2, 2, 1)$		
$c_2 = -1.\ \ \epsilon_1 = \epsilon_2 = 1, \epsilon_3 = 0$		
$F_{18}' : f = (1, 2, 3, -1, -\frac{1}{2}, 0)$	± 1.	3
$\mathfrak{F} = (5, 11/4, 2, 1, 1, \frac{1}{2})$		
$c_2 = -1.\ \ \epsilon_1 = 0, \epsilon_2 = 1, \epsilon_3 = 0$		
$F_{18}'' : f = (1, 1, 6, 0, 0, -\frac{1}{2})$	$\pm 1, \pm j_3, \pm(1 - j_3)$.	3
$\mathfrak{F} = (6, 6, \frac{3}{4}, 0, 0, 3)$		
$c_2 = -1.\ \ \epsilon_1 = \epsilon_2 = 0, \epsilon_3 = 1$		
$F_{18}''' : f = (2, 2, 2, \frac{1}{2}, 1, 1)$	± 1.	2, 3
$\mathfrak{F} = (15/4, 3, 3, 0, -\frac{3}{2}, -\frac{3}{2})$		
$c_3 = -1.\ \ \epsilon_1 = 1, \epsilon_2 = \epsilon_3 = 0$		
$F_{20} : f = (1, 2, 3, -1, 0, 0)$	± 1.	2
$\mathfrak{F} = (5, 3, 2, 1, 0, 0)$		
$c_5 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$		
$F_{22} : f = (1, 2, 3, 0, -\frac{1}{2}, 0)$	± 1.	11
$\mathfrak{F} = (6, 11/4, 2, 0, 1, 0)$		
$c_2 = -1.\ \ \epsilon_1 = \epsilon_3 = 0, \epsilon_2 = 1$		
$F_{24} : f = (2, 2, 2, 0, 0, -1)$	± 1.	2
$\mathfrak{F} = (4, 4, 3, 0, 0, 2)$		
$c_3 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$		

Table I—*Continued*

	Units (in j-coordinates)	All $p \ni p^2 \mid d$ or $p \parallel d$, $c_p = 1$
$F_{25}: f = (2, 2, 2, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ $\mathfrak{F} = \frac{1}{4}(15, 15, 15, 5, 5, 5)$ $c_5 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$	$\pm 1.$	5
$F_{27}: f = (2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ $\mathfrak{F} = \frac{1}{4}(15, 15, 15, -3, -3, -3)$ $c_3 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$	$\pm 1.$	2, 3
$F_{28}: f = (1, 3, 3, 1, \frac{1}{2}, \frac{1}{2})$ $\mathfrak{F} = (8, 11/4, 11/4, -\frac{3}{4}, -1, -1)$ $c_7 = -1.\ \ \epsilon_1 = 0, \epsilon_2 = \epsilon_3 = 1$	$\pm 1.$	2
$F_{32}: f = (2, 2, 2, 0, 0, 0)$ $\mathfrak{F} = (4, 4, 4, 0, 0, 0)$ $c_2 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$	$\pm 1.$	2
$F'_{32}: f = (2, 2, 3, -1, -1, 0)$ $\mathfrak{F} = (5, 5, 4, 2, 2, 1)$ $c_2 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$	$\pm 1.$	2
$F_{36}: f = (2, 2, 3, 0, 0, -1)$ $\mathfrak{F} = (6, 6, 3, 0, 0, 3)$ $c_2 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$	$\pm 1.$	2, 3
$F'_{36}: f = (1, 3, 3, 0, 0, 0)$ $\mathfrak{F} = (9, 3, 3, 0, 0, 0)$ $c_3 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$	$\pm 1.$	2, 3
$F_{48}: f = (1, 4, 4, -2, 0, 0)$ $\mathfrak{F} = (12, 4, 4, 2, 0, 0)$ $c_3 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$	$\pm 1.$	2
$F_{50}: f = (2, 3, 3, \frac{1}{2}, 1, 1)$ $\mathfrak{F} = (35/4, 5, 5, 0, -5/2, -5/2)$ $c_2 = -1.\ \ \epsilon_1 = 1, \epsilon_2 = \epsilon_3 = 0$	$\pm 1.$	5
$F_{54}: f = (2, 3, 3, -\frac{3}{2}, 0, 0)$ $\mathfrak{F} = (27/4, 6, 6, 3, 0, 0)$ $c_2 = -1.\ \ \epsilon_1 = 1, \epsilon_2 = \epsilon_3 = 0$	$\pm 1.$	3
$F'_{54}: f = (3, 3, 3, \frac{3}{2}, \frac{3}{2}, \frac{3}{2})$ $\mathfrak{F} = \frac{1}{4}(27, 27, 27, -9, -9, -9)$ $c_2 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$	$\pm 1.$	3
$F_{64}: f = (3, 3, 3, -1, -1, -1)$ $\mathfrak{F} = (8, 8, 8, 4, 4, 4)$ $c_2 = -1.\ \ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$	$\pm 1.$	2

Each of the 39 norm-forms F represents all positive integers except in the following cases:

F_d	Numbers excluded	F_d	Numbers excluded
F'_8	$4n + 2$	F_{28}	$4n + 2$
F_9	$3n + 2$	F_{32}	$4n + 2$, and $4n + 3$
F'_{12}	$4n + 2$	F'_{32}	$4n + 2$, and $4n + 3$
F'_{16}	$4n + 3$	F_{36}	$3n + 2$
F''_{16}	$4n + 2$	F'_{36}	$3n + 2$
F_{18}	$3(3n \pm 1)$	F_{48}	$4n + 2$, and $4n + 3$
F''_{18}	$3n + 2$	F_{50}	$5n \pm 2$
F'''_{18}	$3n + 2$	F_{54}	$3n + 2$ and $3(3n + 1)$
F_{24}	$4n + 2$	F'_{54}	$3n + 2$ and $3(3n \pm 1)$
F_{25}	$5n \pm 2$	F_{64}	$4n + 2$, $4n + 3$, and $8n + 5$
F_{27}	$3n + 2$ and $3(3n + 1)$		

Table II

	Equivalent diagonal form:	Form of Quaternion in new system:	Integrality conditions: y_i integers and
F_2 :	(1, 1, 1) Units: ± 1, $\pm i_1$, $\pm i_2$, $\pm i_3$ $\pm\frac{1}{2}(1 \pm i_1 \pm i_2 \pm i_3)$	$\frac{1}{2}(1, 1, 1, 1)$	$y_0 \equiv y_1 \equiv y_2 \equiv y_3$, mod 2.
F_3 :	(1, 1, 3) Units: ± 1, $\pm i_3$, $\pm\frac{1}{2}(1 \pm i_1)$, $\pm(i_2 \pm i_3)$	$\frac{1}{2}(1, 1, 1, 1)$	$y_0 \equiv y_1$, $y_2 \equiv y_3$, mod 2.
F_4 :	(1, 1, 1) Units: ± 1, $\pm i_1$, $\pm i_2$, $\pm i_3$	(1, 1, 1, 1)	none
F_5 :	(1, 2, 5) Units: ± 1, $\pm(\pm\frac{1}{2} + \frac{1}{4}i_1 - \frac{1}{4}i_3)$	$\frac{1}{4}(2, 1, 2, 1)$	$\begin{cases} y_1 + y_3 \equiv 2y_2, \text{ mod } 4. \\ y_1 + y_2 \equiv y_0, \text{ mod } 2. \end{cases}$
F_6 :	(1, 1, 3) Units: ± 1, $\pm i_3$	$\frac{1}{2}(1, 1, 1, 1)$	$y_0 \equiv y_1 \equiv y_2 \equiv y_3$, mod 2.
F'_6 :	(1, 1, 1) Units: ± 1, $\pm(\pm\frac{1}{2} + \frac{1}{2}i_1 - \frac{1}{2}i_2 - \frac{1}{2}i_3)$	$\frac{1}{2}(1, 1, 1, 1)$	$\begin{cases} y_0 \equiv y_1 \equiv y_2 \equiv y_3, \text{ mod } 2. \\ y_1 \equiv y_2 + y_3, \text{ mod } 3. \end{cases}$
F_7 :	(1, 1, 7) Units: ± 1, $\pm i_3$.	$\frac{1}{2}(1, 1, 1, 1)$	$y_0 \equiv y_1$, $y_2 \equiv y_3$, mod 2.
F_8 :	(1, 1, 1) Units: ± 1, $\pm i_3$.	(1, 1, 1, 1)	$y_1 \equiv y_2$, mod 2.
F'_8 :	(1, 1, 1) Units: ± 1, $\pm\frac{1}{2}(\pm 1 + i_1 + i_2 + i_3)$	$\frac{1}{2}(1, 1, 1, 1)$	$\begin{cases} y_1 \equiv y_2 \equiv y_3, \text{ mod } 4. \\ y_0 \equiv y_3, \text{ mod } 2. \end{cases}$

Table II—*Continued*

	Equivalent diagonal form:	Form of Quaternion in new system:	Integrality conditions: y_i integers and
F_9:	$(1, 1, 3)$ Units: $\pm 1, \pm\frac{1}{2}(1 \pm i_2)$	$\frac{1}{2}(1, 1, 1, 3)$	$y_0 \equiv y_2, y_1 \equiv y_3$, mod 2.
F_{10}:	$(1, 2, 5)$ Units: ± 1.	$\frac{1}{2}(1, 1, 1, 1)$	$y_0 \equiv y_1 + y_3 \equiv y_2$, mod 2.
F'_{10}:	$(1, 1, 1)$ Units: $\pm 1, \pm i_3$.	$\frac{1}{2}(1, 1, 1, 1)$	$\begin{cases} y_0 \equiv y_1 \equiv y_2 \equiv y_3, \text{ mod } 2. \\ y_2 \equiv 2y_1, \text{ mod } 5. \end{cases}$
F_{12}:	$(1, 1, 3)$ Units: $\pm 1, \pm i_3$	$(1, 1, 1, 1)$	none.
F'_{12}:	$(1, 1, 3)$ Units: $\pm 1, \pm\frac{1}{2}(1 + i_1)$.	$\frac{1}{2}(1, 1, 2, 2)$	$y_0 \equiv y_1, y_2 \equiv y_3$, mod 2.
F''_{12}:	$(1, 1, 3)$ Units: ± 1.	$\frac{1}{2}(1, 1, 2, 2)$ or $\frac{1}{2}(1, 1, 1, 1)$	$y_0 \equiv y_1 \equiv y_2 + y_3$, mod 2. $\begin{cases} y_0 \equiv y_1, \text{ mod } 2. \\ 2y_1 \equiv y_3 - 3y_2, \text{ mod } 8. \end{cases}$
F'''_{12}:	$(1, 1, 1)$ Units: ± 1.	$(1, 1, 1, 1)$	$y_1 + y_2 + y_3 \equiv 0$, mod 3.
F_{13}:	$(1, 2, 13)$ Units: ± 1.	$\frac{1}{4}(2, 1, 2, 1)$	$\begin{cases} y_3 \equiv y_1 + 2y_2, \text{ mod } 4. \\ y_0 \equiv y_1 + y_2, \text{ mod } 2. \end{cases}$
F_{16}:	$(1, 1, 1)$ Units: ± 1.	$(1, 1, 1, 1)$	y_1 even, $y_2 \equiv y_3$, mod 2.
F'_{16}:	$(1, 1, 1)$ Units: $\pm 1, \pm i_3$.	$(1, 1, 1, 1)$	y_1, y_2 even.
F''_{16}:	$(1, 1, 1)$ Units: ± 1.	$(1, 1, 1, 1)$	$y_1 \equiv y_2 \equiv y_3$, mod 2.
F_{18}:	$(1, 1, 1)$ Units: $\pm 1, \pm i_3$.	$\frac{1}{2}(1, 3, 3, 1)$	$y_0 \equiv y_1 \equiv y_2 \equiv y_3$, mod 2.
F'_{18}:	$(1, 1, 1)$ Units: ± 1.	$\frac{1}{2}(1, 1, 1, 1)$	$\begin{cases} y_0 \equiv y_1 \equiv y_2 \equiv y_3, \text{ mod } 2. \\ 4y_1 + y_2 + y_3 \equiv 0, \text{ mod } 9. \end{cases}$
F''_{18}:	$(1, 1, 1)$ Units: ± 1, $\pm\frac{1}{2}(\pm 1 + i_1 + i_2 - i_3)$	$\frac{1}{2}(1, 1, 1, 1)$	$\begin{cases} y_0 \equiv y_1 \equiv y_2 \equiv y_3, \text{ mod } 2. \\ y_1 \equiv y_2 \equiv -y_3, \text{ mod } 3. \end{cases}$
F'''_{18}:	$(1, 1, 3)$ Units: ± 1.	$\frac{1}{2}(1, 1, 1, 3)$ or $\frac{1}{2}(1, 1, 1, 3)$	$\begin{cases} y_0 \equiv y_1, y_2 \equiv y_3, \text{ mod } 2. \\ 2y_1 \equiv y_2 + y_3, \text{ mod } 4. \end{cases}$ $y_0 \equiv y_1 \equiv y_2 \equiv y_3$, mod 2.
F_{20}:	$(1, 2, 5)$ Units: ± 1.	$\frac{1}{2}(2, 1, 2, 1)$	$y_1 \equiv y_3$, mod 2.
F_{22}:	$(1, 1, 1)$ Units: ± 1.	$\frac{1}{2}(1, 1, 1, 1)$	$\begin{cases} y_0 \equiv y_1 \equiv y_2 \equiv y_3, \text{ mod } 2. \\ y_1 + y_3 \equiv 3y_2, \text{ mod } 11. \end{cases}$
F_{24}:	$(1, 1, 3)$ Units: ± 1.	$(1, 1, 1, 1)$	$y_1 \equiv y_3$, mod 2.

Table II—*Continued.*

	Equivalent diagonal form:	Form of Quaternion in new system:	Integrality conditions: y_i integers and
F_{25}:	(1, 2, 5) Units: ± 1.	$\frac{1}{4}(2, 1, 2, 5)$	$\begin{cases} y_1 - y_3 \equiv 2y_2, \text{ mod } 4. \\ y_0 - y_3 \equiv y_2, \text{ mod } 2. \end{cases}$
F_{27}:	(1, 1, 3) Units: ± 1.	$\frac{1}{2}(1, 1, 1, 3)$	$\begin{cases} y_1 \equiv -y_2, \text{ mod } 3. \\ y_0 \equiv y_2, \ y_1 \equiv y_3, \text{ mod } 2. \end{cases}$
F_{28}:	(1, 1, 7) Units: ± 1.	$\frac{1}{2}(1, 1, 2, 2)$	$y_0 \equiv y_1 \equiv y_2 + y_3$, mod 2.
F_{32}:	(1, 1, 1) Units: ± 1.	(1, 1, 1, 1)	$y_1 \equiv y_2 \equiv y_3 \equiv 0$, mod 2.
F'_{32}:	(1, 1, 1) Units: ± 1.	(1, 1, 1, 1)	$\begin{cases} y_1 \equiv y_2 \equiv 0, \text{ mod } 2. \\ 2y_3 \equiv y_1 + y_2, \text{ mod } 4. \end{cases}$
F_{36}:	(1, 1, 1) Units: ± 1.	(1, 1, 1, 1)	$y_1 \equiv y_2 \equiv y_3$, mod 3.
F'_{36}:	(1, 1, 3) Units: ± 1.	(1, 1, 1, 1)	$y_3 \equiv 0$, mod 3.
F_{48}:	(1, 1, 3) Units: ± 1.	(1, 2, 1, 1)	$y_2 \equiv y_3$, mod 2.
F_{50}:	(1, 1, 1) Units: ± 1.	$\frac{1}{2}(1, 5, 1, 1)$	$\begin{cases} y_0 \equiv y_1 \equiv y_2 \equiv y_3, \text{ mod } 2. \\ 2y_2 \equiv y_3, \text{ mod } 5. \end{cases}$
F_{54}:	(1, 1, 1) Units: ± 1.	$\frac{1}{2}(1, 1, 1, 1)$	$\begin{cases} y_0 \equiv y_1 \equiv y_2 \equiv y_3, \text{ mod } 2. \\ y_1 \equiv y_2 \equiv y_3, \text{ mod } 3; \\ y_1 + y_2 + y_3 \equiv 0, \text{ mod } 9. \end{cases}$
F'_{54}:	(1, 1, 1) Units: ± 1.	$\frac{1}{2}(1, 1, 1, 1)$	$\begin{cases} y_0 \equiv y_1 \equiv y_2 \equiv y_3, \text{ mod } 2, \\ 4y_1 + y_2 + y_3 \equiv 0, \text{ mod } 9, \\ y_1 \equiv y_2, \text{ mod } 3. \end{cases}$
		or $\frac{1}{2}(1, 3, 3, 3)$	$y_0 \equiv y_1 \equiv y_2 \equiv y_3$, mod 2.
F_{64}:	(1, 1, 1) Units: ± 1.	(1, 2, 2, 2)	$y_1 + y_2 + y_3 \equiv 0$, mod 2.

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