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LEVELS OF POSITIVE DEFINITE TERNARY QUADRATIC FORMS

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ABSTRACT. The level N and squarefree character q of a positive definite ternary quadratic form are defined so that its associated modular form has level N and character χ_q. We define a collection of correspondences between classes of quadratic forms having the same level and different discriminants. This makes practical a method for finding representatives of all classes of ternary forms having a given level. We also give a formula for the number of genera of ternary forms with a given level and character.

INTRODUCTION

In this article, we consider some questions concerning the classification of positive definite ternary quadratic forms. Our motivation is the connection between quadratic forms and modular forms which is given in the theorem below. We first recall some notation and terminology concerning modular forms.

Define a symbol (a/b) for $a, b \in \mathbb{Z}$ by the following conditions:

(1) (a/b) is the Legendre symbol if b is an odd prime.
(2) $(a/2) = (-1)^{(a^2-1)/8}$ if a is odd.
(3) $(a/-1) = 1$ if $a \geq 0$, $(a/-1) = -1$ if $a < 0$.
(4) $(a/b) = 0$ if $\gcd(a, b) > 1$, $(1/0) = 1$, $(a/0) = 0$ if $a \neq 1$.
(5) $(a/bc) = (a/b) \cdot (a/c)$ for all $b, c \in \mathbb{Z}$.

If t is a nonzero integer, define a function χ_t on the integers as follows: Let $t = qr^2$ with q squarefree. If $q \equiv 1 \pmod 4$, let $D = q$. If $q \equiv 2, 3 \pmod 4$, let $D = 4q$. Then $\chi_t(n) = (D/n)$ for all $n \in \mathbb{Z}$. The function χ_t is a quadratic Dirichlet character with conductor $|D|$ [11].

Let k be an integer, N a positive integer (divisible by 4 if k is odd), and χ a character modulo N. Let $\Gamma_0(N)$ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of all $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $c \equiv 0 \pmod N$. A modular form θ is said to have *weight* $k/2$, *level* N, and *character* χ if for all $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$ and all $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$,

$$\theta\left(\frac{az+b}{cz+d}\right) = \begin{cases} \chi(d) \cdot (cz+d)^{k/2} \cdot \theta(z) & \text{if } k \text{ is even}, \\ \chi(d) \cdot j(\gamma, z)^k \cdot \theta(z) & \text{if } k \text{ is odd}. \end{cases}$$

Here, $j(\gamma, z) = \varepsilon_d^{-1} \chi_c(d)(cz+d)^{1/2}$, where $\varepsilon_d = 1$ or i as $d \equiv 1$ or $3 \pmod 4$. Denote the vector space of all such modular forms as $M_{k/2}(N, \chi)$,

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and its subspace of cusp forms as $S_{k/2}(N, \chi)$. (See [12] or [7] for more background on modular forms, particularly those of half-integral weight.)

Theorem (Shimura [12]). *Let $f(x_1, \dots, x_n)$ be a positive definite quadratic form having integer coefficients. Let A be the $n \times n$ matrix*

$$A = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right].$$

Define N to be the smallest positive integer so that NA^{-1} is an even matrix, that is, has integral entries, and even integers on the main diagonal. Let $\theta(f) = \theta_f(z)$ be defined by

$$\theta_f(z) = \sum q^{f(m_1, \dots, m_n)},$$

where $q = e^{2\pi i z}$, and the sum is taken over all n-tuples $(m_1, \dots, m_n)$ in $\mathbb{Z}^n$. Then $\theta(f) \in M_{n/2}(N, \chi_d)$, where $d = \det(A)$ if $n \equiv 0 \pmod 4$, $d = -\det(A)$ if $n \equiv 2 \pmod 4$, and $d = \det(A)/2$ if n is odd.

Remarks. This theorem is a special case of Proposition 2.1 in [12]. Shimura's proposition generalizes results of Hecke and Schoeneberg in the case when n is even, and of Pfetzer when n is odd (see [12] for references). It is not hard to see that $\det(A)$ is even if n is odd, so the discriminant d of f is an integer in each case. In saying that χ_d is the character of $\theta(f)$, we mean that $\theta(f)$ has character χ such that $\chi(a) = \chi_d(a)$ if $\gcd(a, N) = 1$. (By definition, $\chi(a) = 0$ if $\gcd(a, N) > 1$.) Suppose that $g = cf$, with c a positive integer. Then $\theta(g)$ has weight $n/2$ and level cN. Its character is χ_d if n is even, χ_{cd} if n is odd. As a power series in q, $\theta(g)$ is the same as $\theta(f)$ with all exponents multiplied by c. So we can restrict our attention to the case where f is primitive, that is, where the greatest common divisor of the coefficients of f is 1. Finally, we have that if f_1 and f_2 are in the same genus of forms (see §3), then $\theta(f_1) - \theta(f_2) \in S_{n/2}(N, \chi_d)$ [10].

Attempts have been made to use quadratic forms to describe a space of modular or cusp forms of a given weight, level, and character. In formal terms, this can be considered as a special case of the "basis problem," which was successfully dealt with in [5] in the case in which the weight is an integer $k \geq 2$. Serre and Stark [11] found bases for all spaces of forms of weight 1/2, using theta series, which may be defined in terms of quadratic forms. In [8], the author employed quadratic forms to construct a basis for $S_{3/2}(196, \chi_7)$, in order to fully compute the effects of the Hecke operators on this space, and the Shimura correspondence on associated eigenforms. Obviously, it would be helpful in this application to be able to find all primitive quadratic forms which lead to a particular value of the level N. If $n = 1$, then this is trivial, as there is only one primitive form in that case. In the case of binary forms $(n = 2)$, this problem is the same as that of finding all primitive forms of a given discriminant. For if $f(x_1, x_2) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$ is primitive, then

$$A = \begin{bmatrix} 2a_{11} & a_{12} \\ a_{12} & 2a_{22} \end{bmatrix} \quad \text{and} \quad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 2a_{22} & -a_{12} \\ -a_{12} & 2a_{11} \end{bmatrix},$$

leading to the conclusion that $N = \det(A)$ in every case.

When we look at ternary forms $(n = 3)$, however, this is no longer the case. For example, let

$$f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 8x_3^2$$

and

$$g(x_1, x_2, x_3) = 3x_1^2 + 11x_2^2 + 11x_3^2 - 10x_2x_3 - 2x_1x_3 - 2x_1x_2.$$

Then $\theta(f)$ and $\theta(g)$ are both weight 3/2 forms of level 32 and trivial character. But f has discriminant 64 while that of g is 1024.

This example illustrates another point. Extensive tables of positive definite ternary quadratic forms, grouped by discriminant, have been compiled. In particular, the tables of Brandt and Intrau [1] list (in over 200 pages) all reduced ternary forms with $d \leq 1000$. But, as we see above, a modular form of relatively small level may arise from a quadratic form with a large discriminant.

In this article, we will consider the following question: Is it possible to find all primitive, positive definite, ternary quadratic forms whose associated modular forms possess a particular level? We will show that this is possible in general, and illustrate a practical method for doing so for a large number of values of the level.

1. Ternary quadratic forms

The literature on quadratic forms is extensive and highly developed. We will take an elementary approach to the subject, focusing narrowly on positive definite ternary quadratic forms which are defined over the integers. However, our approach is unique in that it stresses the level throughout as the invariant of importance for a quadratic form.

Let f be a ternary quadratic form with integer coefficients, given by the equation

$$f(x, y, z) = ax^2 + by^2 + cz^2 + ryz + sxz + txy. \tag{1}$$

Unless otherwise stated, we assume that f is *positive definite* (that is, that $f(x, y, z) > 0$ for real numbers x, y, z unless $x = y = z = 0$) and *primitive* ($\gcd(a, b, c, r, s, t) = 1$). (Note that we do not follow the "classically integral" definition, which requires that r, s, and t be even integers. Some results quoted below, particularly those of Dickson [4], have been restated to account for this difference in definitions.) We will also denote f by the array $f = \left(\begin{smallmatrix} a & b & c \\ r & s & t \end{smallmatrix}\right)$.

Define the *matrix* of f to be

$$A = A_f = \begin{bmatrix} 2a & t & s \\ t & 2b & r \\ s & r & 2c \end{bmatrix}.$$

We will say that a 3×3 matrix is *primitive* if it is the matrix of a primitive ternary form. Define the *discriminant* of f to be

$$d = d_f = \frac{\det(A)}{2} = 4abc + rst - ar^2 - bs^2 - ct^2.$$

Let A_{ij} be the i, j-cofactor of A. That is,

$$\begin{aligned} A_{11} &= 4bc - r^2, & A_{23} &= st - 2ar = A_{32}, \\ A_{22} &= 4ac - s^2, & A_{13} &= rt - 2bs = A_{31}, \\ A_{33} &= 4ab - t^2, & A_{12} &= rs - 2ct = A_{21}. \end{aligned}$$

Define the *divisor* of f to be the positive integer

$$m = m_f = \gcd(A_{11}, A_{22}, A_{33}, 2A_{23}, 2A_{13}, 2A_{12}).$$

Let $\alpha = A_{11}/m$, $\beta = A_{22}/m$, $\gamma = A_{33}/m$, $\rho = 2A_{23}/m$, $\sigma = 2A_{13}/m$, and $\tau = 2A_{12}/m$. Define the *reciprocal* of f to be the ternary form

$$\phi(x, y, z) = \alpha x^2 + \beta y^2 + \gamma z^2 + \rho yz + \sigma xz + \tau xy. \tag{2}$$

It is clear that ϕ is a primitive positive definite form.

The matrix of ϕ is

$$A_\phi = \begin{bmatrix} 2\alpha & \tau & \sigma \\ \tau & 2\beta & \rho \\ \sigma & \rho & 2\gamma \end{bmatrix} = \frac{2}{m}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \frac{2\det(A)}{m} A_f^{-1}$$

by the usual cofactor results. So $A_\phi = \frac{4d}{m} A_f^{-1}$. Notice that m divides $4d$ because

$$4d = 2\det(A) = 4a(A_{11}) + t(2A_{12}) + s(2A_{13}).$$

Define the *level* of f to be the positive integer $N = N_f = 4d_f/m_f$. Note that, as in the introduction, N is the smallest positive integer such that NA_f^{-1} is even. We can also describe the level of f as the unique positive integer N so that NA_f^{-1} is a primitive matrix.

Now consider the definitions above applied to the primitive form ϕ. Since $A_\phi = N_f A_f^{-1}$, the discriminant of ϕ is

$$d_\phi = \frac{\det(A_\phi)}{2} = \frac{N_f^3}{2}\det(A_f^{-1}) = \frac{N_f^3}{4d_f}.$$

Let m_ϕ be the divisor of ϕ, and let $N_\phi = 4d_\phi/m_\phi$ be its level. Let F be the reciprocal of ϕ. Then

$$A_F = N_\phi A_\phi^{-1} = N_\phi (N_f A_f^{-1})^{-1} = \frac{N_\phi}{N_f} A_f.$$

But F is a primitive form by the definition of the reciprocal. So A_F is a primitive matrix, as is A_f. Clearly, the only way in which a positive scalar multiple of a primitive matrix can be primitive is if the scalar is 1. Therefore, f is the reciprocal of ϕ. Furthermore, we have the following important fact.

Theorem 1. *Let f be a primitive, positive definite, ternary quadratic form, and let ϕ be its reciprocal. Then f and ϕ have the same level.*

Fix the following notation now. Considered as constants depending on f, denote d_f by d, m_f by m, d_ϕ by δ, m_ϕ by μ, and the common value of N_f and N_ϕ by N. Each of these quantities is a positive integer.

Ternary forms f and g are said to be *equivalent*, $f \sim g$, if there is a unimodular matrix $U = [u_{ij}]$ so that $A_g = UA_fU^t$. (That is, U has integer entries and $\det(U) = \pm 1$; U^t is its transpose.) In this case, the coefficients of g can be expressed explicitly in terms of f as follows. For $i = 1, 2, 3$, let $\mathbf{u}_i = (u_{i1}, u_{i2}, u_{i3})$. Suppose that $g(x_1, x_2, x_3) = \sum_{i\leq j} a_{ij}x_ix_j$. Then

$$a_{ij} = \begin{cases} f(\mathbf{u}_i) & \text{if } i = j, \\ f(\mathbf{u}_i + \mathbf{u}_j) - f(\mathbf{u}_i) - f(\mathbf{u}_j) & \text{if } i \neq j. \end{cases} \tag{3}$$

Equivalent forms are said to belong to the same *class*. Clearly, if $f \sim g$, then $d_f = d_g$.

Proposition 1. *The level of a form is a class invariant. That is, if* $f \sim g$, *then* $N_f = N_g$.

Proof. If some prime divides each coefficient of f, then by equation (3) it divides each coefficient of g. It follows that f is primitive if and only if g is primitive. Now $N_f A_g^{-1} = V A_\phi V^t$, where $V = (U^t)^{-1}$ is unimodular. Since ϕ is a primitive form, $N_f A_g^{-1}$ must be a primitive matrix. But N_g is the unique positive integer so that $N_g A_g^{-1}$ is primitive. Therefore $N_f = N_g$. □

Corollary 1. *If* f *and* g *are equivalent, then their reciprocals are equivalent as well.*

From the equations $\delta = N^3/4d$, $m = 4d/N$, and $\mu = 4\delta/N = N^2/d$, we see that m, μ, and δ are also class invariants. Notice also that $m\mu = 4N$, $m^2\mu = 16d$, and $m\mu^2 = 16\delta$. From these latter two equations we can see that if m is odd, then $16 \mid \mu$, and if μ is odd, then $16 \mid m$. But m is odd if and only if one of A_{11}, A_{22}, A_{33} is odd. This is the case if and only if one of r, s, t is odd. If not, then it is easy to see that $4 \mid m$. Similarly, either μ is odd or $4 \mid \mu$. In any case, we see that $16 \mid m\mu$ and thus that $4 \mid N$.

Now suppose that f is a ternary form having a given level N. What can be concluded about the discriminant d of such a form? First note that if $p \mid d$, then $p \mid \mu d$, so $p \mid N^2$ and $p \mid N$. So d cannot be divisible by any prime which does not divide N. Let p be an odd prime and suppose that $p^g \parallel N$ (that is, N is divisible by p^g, but not by p^{g+1}). Suppose that $p^h \parallel d$. From the fact that $m = 4d/N$ and $\mu = N^2/d$ are integers, we see that $g \le h \le 2g$. Suppose that $2^g \parallel N$ (so that $g \ge 2$) and that $2^h \parallel d$. We can now conclude that $h + 2 \ge g$ and $2g \ge h$, that is, $g - 2 \le h \le 2g$. But as noted in the previous paragraph, m is either odd or divisible by 4, and likewise for μ. So we see that $h \ne g-1$ and $h \ne 2g-1$ in this case. (If f is a quadratic form in an even number of variables, then it is known that N and d are divisible by the same prime factors [9]. Note that for ternary forms, we may have d odd although N is even.)

There is an additional restriction on discriminant values. First note the following result which we will use on several occasions.

Proposition 2 [4, pp. 12–17]. *Let* f *be a ternary form. Let* m *be its divisor and* μ *be the divisor of its reciprocal. Then* f *is equivalent to a form* $\begin{pmatrix} a & b & c \\ r & s & t \end{pmatrix}$, *having reciprocal* $\begin{pmatrix} \alpha & \beta & \gamma \\ \rho & \sigma & \tau \end{pmatrix}$, *so that* a *and* γ *are relatively prime to each other and to* $m\mu$.

Lemma 1. *There is no primitive ternary form* f *with divisor* m, *whose reciprocal has divisor* μ, *so that* m *and* μ *are both squares and either* m *or* μ *is odd.*

Proof. Suppose that f is such a form. We may assume that f and its reciprocal are as given in Proposition 2. In particular then, a and γ are odd and positive, and so we may consider the Jacobi symbols $(m\gamma/a)$ and $(\mu a/\gamma)$. By the definition of the reciprocal, we have that $m\gamma = 4ab - t^2$ and $\mu a = 4\beta\gamma - \rho^2$. Since m and μ are squares, it follows that

$$\left(\frac{\gamma}{a}\right) = \left(\frac{m\gamma}{a}\right) = \left(\frac{4ab - t^2}{a}\right) = \left(\frac{-1}{a}\right)$$

and

$$\left(\frac{a}{\gamma}\right) = \left(\frac{\mu a}{\gamma}\right) = \left(\frac{4\beta\gamma - \rho^2}{\gamma}\right) = \left(\frac{-1}{\gamma}\right).$$

So

$$\left(\frac{\gamma}{a}\right)\left(\frac{a}{\gamma}\right) = \left(\frac{-1}{a}\right)\left(\frac{-1}{\gamma}\right),$$

and by Quadratic Reciprocity,

$$(-1)^{(a-1)(\gamma-1)/4} = (-1)^{(a-1)/2}(-1)^{(\gamma-1)/2}.$$

But if m is odd (and a square), then $\gamma \equiv m\gamma \equiv 4ab - t^2 \equiv -1 \pmod 4$. Then it follows that

$$(-1)^{(a-1)/2} = -(-1)^{(a-1)/2},$$

which is impossible. There is a similar contradiction if μ is odd. So f cannot exist under these conditions. □

The divisors m and μ are both squares if and only if $N = m\mu/4$ and $d = mN/4$ are both squares. We summarize the above results as:

Theorem 2. *Let f be a primitive, positive definite, ternary quadratic form with level N and discriminant d. Suppose that*

$$N = 2^{n_0} p_1^{n_1} \dots p_k^{n_k} \tag{4}$$

is the prime factorization of N. Then $n_0 \geq 2$ and d is of the form

$$d = 2^{d_0} p_1^{d_1} \dots p_k^{d_k} \tag{5}$$

with the following restrictions on exponents:

(1) *$d_0 = n_0 - 2$, $d_0 = 2n_0$, or $n_0 \leq d_0 \leq 2n_0 - 2$, and*
(2) *for $1 \leq i \leq k$, $n_i \leq d_i \leq 2n_i$.*

Furthermore, if n_i is even for $0 \leq i \leq k$, then either $n_0 \leq d_0 \leq 2n_0 - 2$, or d_i is odd for some $1 \leq i \leq k$.

In particular, we see that given a value N, there is only a finite number of values d so that a ternary form could have level N and discriminant d. These values are explicitly calculable in terms of N. (In the following sections, when we write that N and d are given by equations (4) and (5), we will assume that they satisfy the conditions on exponents which are given in Theorem 2. In §3, we will see that there is in fact a ternary form for every level N and discriminant d which are allowed by this theorem.)

2. Construction of all forms of a given level

Given a value d, it is possible (in theory) to find a representative of each class of primitive, positive definite, ternary quadratic forms having discriminant d. We sketch the method here.

Proposition 3 [4, pp. 155–179]. *Let f be a ternary form given by equation (1). Say that f is reduced if the following are true:*

(1) $a \leq b \leq c$;
(2) *r, s, and t are all positive or all nonpositive;*
(3) $a \geq |t|$; $a \geq |s|$; $b \geq |r|$;

(4) $a+b+r+s+t \geq 0$;
(5) $a=t \Rightarrow s \leq 2r$; $a=s \Rightarrow t \leq 2r$; $b=r \Rightarrow t \leq 2s$;
(6) $a=-t \Rightarrow s=0$; $a=-s \Rightarrow t=0$; $b=-r \Rightarrow t=0$;
(7) $a+b+r+s+t=0 \Rightarrow 2a+2s+t \leq 0$;
(8) $a=b \Rightarrow |r| \leq |s|$; $b=c \Rightarrow |s| \leq |t|$.

Then every primitive, positive definite, ternary form is equivalent to one and only one reduced form. Also, if f is reduced and has discriminant d, then $d/4 \leq abc \leq d/2$.

Remark. The above criteria for a reduced form were first provided by Eisenstein. Other definitions are possible. In particular, a form which is reduced by this definition is not necessarily "Minkowski reduced," a definition which requires in part that $r+s+t \leq a+b$ in all cases [3, p. 396].

This means that there is only a finite number of possibilities for the coefficients of a reduced form having a given discriminant. In particular, if f is a reduced form, with discriminant d, given by equation (1), then

$$1 \leq a \leq \sqrt[3]{d/2}, \quad a \leq b \leq \sqrt{d/2a}, \quad \max(b, d/4ab) \leq c \leq d/2ab,$$

and either

$$-b \leq r \leq 0, \quad -a \leq s \leq 0, \quad -a \leq t \leq 0,$$

or

$$1 \leq r \leq b, \quad 1 \leq s \leq a, \quad 1 \leq t \leq a.$$

Starting with a given value of N, we could use Theorem 2 to find the finite collection of potential discriminants for forms of level N. For each such discriminant d, the finite collection of possible coefficients could be tested. Thus, it is theoretically possible to find all reduced forms of a given level. Of course, the larger d is, the more potential coefficients have to be tested. Since for a given N, a corresponding d might be as large as N^2, this direct method could become unworkable for a relatively small value of the level. However, we will note several results which allow us to restrict this search process, thus making it much more practical.

First note that, in some cases, we can place additional restrictions on the potential coefficients of forms, owing to the fact that we want only forms of a specific level. Suppose that f, given by equation (1), is a reduced form of level N and discriminant d, having reciprocal ϕ as in equation (2). Let $m = 4d/N$ and $\mu = N^2/d$ be the respective divisors of f and ϕ. Then:

(1) If m is even, then r, s, and t must also be even. This is because m is the greatest common divisor of a collection of integers including $4bc - r^2$, $4ac - s^2$, and $4ab - t^2$.

(2) If μ is odd, then there are restrictions on the coefficients a, b, and c. We know that $\mu a = 4\beta\gamma - \rho^2$. If a is even, then ρ is even, and so $4 \mid \mu a$ and $4 \mid a$. If a is odd, then ρ is odd, and $\mu a \equiv -\rho^2 \equiv -1 \pmod 4$. So either $a \equiv 0$ or $a \equiv -\mu \pmod 4$. The same is true with b or c in place of a.

(3) Since $m \leq m\gamma = 4ab - t^2 \leq 4ab$, we have that $b \geq m/4a$. (It is worth noting that then we have $c \leq d/2ab \leq 2d/m = N/2$. Thus $N/2$ is an upper limit for the absolute values of the coefficients of a reduced form of level N, independent of its discriminant.)

More importantly though, we may use a collection of functions between classes of forms having a given level to cut down on the number of discriminants for which this coefficient-testing process must be carried out. Let $C(N, d)$ denote the set of all classes of positive definite ternary forms having level N and discriminant d. If f is a ternary form, let $\overline{f}$ denote the class to which f belongs. Our next theorem restates some earlier results (Theorem 1 and Corollary 1).

Theorem 3. *There is a one-to-one correspondence between the sets* $C(N, d)$ *and* $C(N, \delta)$, *where* $\delta = N^3/4d$. *This correspondence is provided by the mapping* $\overline{f} \mapsto \overline{\phi}$, *where* ϕ *is the reciprocal of* f.

Since d and δ are inversely related, we can immediately eliminate the discriminants associated with N for which $d > \frac{1}{2}\sqrt{N^3}$. The following result allows us to restrict our attention further.

Theorem 4. *Let* N *and* d *be given by equations* (4) *and* (5). *Suppose that* $p^g \parallel N$ *and* $p^h \parallel d$ *for some odd prime* p. *Write* d *as* $p^h d'$. *Then there is a one-to-one correspondence between* $C(N, p^h d')$ *and* $C(N, p^{3g-h} d')$.

Before we describe this correspondence, we need the following lemma.

Lemma 2. *Let* f *be a positive definite, primitive ternary form with level* N *and divisor* m. *Suppose that* $p^i \parallel N$ *and* $p^j \parallel m$ *for some odd prime* p *and positive integer* i. *Then* f *is equivalent to a form* $\left(\begin{smallmatrix} a & b & c \\ r & s & t \end{smallmatrix}\right)$ *with* $p^i \parallel a$, $p^i \mid s$ *and* t, $p^j \mid b$ *and* r, *and* $p \nmid c$. *If* $0 < j < i$, *then we can assume that* $p^j \parallel b$.

Proof. By Proposition 2, we may assume at the start that f has reciprocal $\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ \rho & \sigma & \tau \end{smallmatrix}\right)$ with γ not divisible by p. Let $g = \gcd(\sigma, \rho, 2\gamma)$, so that $p \nmid g$. We can form a unimodular matrix U whose first row is $[\sigma/g \quad \rho/g \quad 2\gamma/g]$. Let A be the matrix of f. Then the first row of UA is $[0 \quad 0 \quad N/g]$. (This can be seen from the fact that the third row of A^{-1} is $[\sigma/N \quad \rho/N \quad 2\gamma/N]$.) So then, if $U = [u_{ij}]$, the first row of UAU^t is $[2\gamma N/g^2 \quad u_{23}N/g \quad u_{33}N/g]$. But UAU^t is the matrix of a form $\left(\begin{smallmatrix} a & b & c \\ r & s & t \end{smallmatrix}\right)$ which is equivalent to f. We can see that $a = \gamma N/g^2$, $s = u_{33}N/g$, and $t = u_{23}N/g$. Since $p^i \parallel N$, $p \nmid g$, and $p \nmid \gamma$, it follows that $p^i \parallel a$ and that $p^i \mid s, t$.

Since f is primitive, one of the remaining coefficients must be relatively prime to p. It is easy to see that

$$\begin{pmatrix} a & b & c \\ r & s & t \end{pmatrix} \sim \begin{pmatrix} a & c & b \\ r & t & s \end{pmatrix} \sim \begin{pmatrix} a & b+c+r & c \\ 2c+r & s & s+t \end{pmatrix},$$

so we may assume that either b or c is not divisible by p without affecting the previous results concerning a, s, and t. If $j = 0$, assume that $p \nmid c$. The proof is complete in that case.

If $j > 0$, assume that $p \nmid b$. Let $g = \gcd(-r, 2b)$, so that $p \nmid g$. We can form a unimodular matrix

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -r/g & 2b/g \\ 0 & u & v \end{bmatrix}$$

for some integers u and v. If A is the matrix of $\begin{pmatrix} a & b & c \\ r & s & t \end{pmatrix}$, then UAU^t is the matrix of $\begin{pmatrix} a' & b' & c' \\ r' & s' & t' \end{pmatrix}$ with $a' = a$, $s' = sv + tu$, and $t' = s(2b/g) + t(-r/g)$ (so that our previous results hold for a', s', and t'), and with

$$b' = b(4bc - r^2)/g^2 \quad \text{and} \quad r' = v(4bc - r^2)/g.$$

The divisor m is the greatest common divisor of a collection of integers which includes $4bc - r^2$, and it is easy to see that each of the other integers is divisible by p^i. So we see that $p^j \mid b', r'$, and if $j < i$, then $p^j \parallel b'$. Since f is primitive, p cannot divide c'. So the proof of Lemma 2 is complete. □

Proof of Theorem 4. Let $\overline{f} \in C(N, p^h d')$. Since $m = 4d/N$, we may, by Lemma 2, assume that

$$f = \begin{pmatrix} p^g a & p^{h-g} b & c \\ p^{h-g} r & p^g s & p^g t \end{pmatrix}$$

with a, b, c, r, s, and t integers, $p \nmid ac$. Let

$$f_p = \begin{pmatrix} a & p^{2g-h} b & p^g c \\ p^g r & p^g s & p^{2g-h} t \end{pmatrix}.$$

Notice that f_p is a primitive form. We will show that $\psi\colon C(N, p^h d') \to C(N, p^{3g-h} d')$ defined by $\psi(\overline{f}) = \overline{f_p}$ is a one-to-one correspondence. (We may also denote ψ by ψ_p or ψ^h.)

If A_f and A_{f_p} are the matrices of f and f_p, respectively, then

$$A_{f_p} = PA_f P, \quad \text{where } P = \begin{bmatrix} p^{-g/2} & 0 & 0 \\ 0 & p^{(3g-2h)/2} & 0 \\ 0 & 0 & p^{g/2} \end{bmatrix}.$$

So

$$d_{f_p} = d_f \det(P)^2 = p^h d' p^{3g-2h} = p^{3g-h} d'.$$

To show that $N_{f_p} = N_f$, note that $N_f A_{f_p}^{-1} = P^{-1} A_\phi P^{-1}$, where ϕ is the reciprocal of f. With f as given, it is not hard to see that

$$\phi = \begin{pmatrix} \alpha & p^{2g-h}\beta & p^g \gamma \\ p^g \rho & p^g \sigma & p^{2g-h} \tau \end{pmatrix},$$

with α, β, γ, ρ, σ, and τ integers, $p \nmid \alpha\beta\gamma$. Then $P^{-1} A_\phi P^{-1}$ is the matrix of

$$\begin{pmatrix} p^g \alpha & p^{h-g} \beta & \gamma \\ p^{h-g} \rho & p^g \sigma & p^g \tau \end{pmatrix},$$

which must be primitive. So $N_f A_{f_p}^{-1}$ is primitive, and $N_{f_p} = N_f$ by definition. Thus $\overline{f_p}$ is an element of $C(N, p^{3g-h} d')$.

Next we show that ψ is a well-defined function. Suppose that f and F are representatives of the same class in $C(N, p^h d')$, and that

$$f = \begin{pmatrix} p^g a & p^{h-g} b & c \\ p^{h-g} r & p^g s & p^g t \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} p^g A & p^{h-g} B & C \\ p^{h-g} R & p^g S & p^g T \end{pmatrix},$$

with $p \nmid acAC$, and $p \nmid bB$ if $g < h < 2g$. So $A_F = UA_f U^t$ for some unimodular matrix U. We want to show that $f_p \sim F_p$. We know that $A_{F_p} =$

PA_FP and $A_{f_p} = PA_fP$, so $A_{F_p} = (PUP^{-1})A_{f_p}(PUP^{-1})^t$ (since $P = P^t$). Clearly, $\det(PUP^{-1}) = \det(U) = \pm 1$. If we can show that the entries of PUP^{-1} are integers, then it is unimodular, and f_p and F_p are equivalent.

If $U = [u_{ij}]$, then one can see that

$$PUP^{-1} = \begin{bmatrix} u_{11} & u_{12}p^{h-2g} & u_{13}p^{-g} \\ u_{21}p^{2g-h} & u_{22} & u_{23}p^{g-h} \\ u_{31}p^g & u_{32}p^{h-g} & u_{33} \end{bmatrix},$$

so we would like to show that $p^{2g-h} \mid u_{12}$, $p^g \mid u_{13}$, and $p^{h-g} \mid u_{23}$. This can be seen by looking closely at the implications of the equation $A_F = UA_fU^t$, as given in (3) above. For example, we have that $C = f(u_{31}, u_{32}, u_{33})$, so that $C \equiv cu_{33}^2 \pmod{p^{h-g}}$. If $h > g$, it follows that $p \nmid u_{33}$. Then one can show that

$$p^{h-g}R \equiv 2cu_{33}u_{23} \quad \text{and} \quad p^gS \equiv 2cu_{33}u_{13} \pmod{p^{h-g}}.$$

So $p^{h-g} \mid u_{23}$ and $p^{h-g} \mid u_{13}$. If $h = 2g$, then this is all that we need to show.

Suppose that $2g > h \geq \frac{3}{2}g$. We then have that

$$p^{h-g}B \equiv p^{h-g}bu_{22}^2 + cu_{23}^2 + p^{h-g}ru_{22}u_{23} \equiv p^{h-g}bu_{22}^2 \pmod{p^g},$$

since $2(h-g) \geq g$. Since in this case $p \nmid bB$, it follows that $p \nmid u_{22}$. Now

$$p^gT \equiv 2p^{h-g}bu_{22}u_{12} \pmod{p^g},$$

so we see that $p^{2g-h} \mid u_{12}$. Finally,

$$p^gS \equiv 2cu_{33}u_{13} \pmod{p^g},$$

so $p^g \mid u_{13}$. So PUP^{-1} is unimodular if $2g \geq h \geq \frac{3}{2}g$.

Now if $h < \frac{3}{2}g$, consider the reciprocals of f and F, say ϕ and Φ. If $A_F = UA_fU^t$, then $A_\Phi = VA_\phi V^t$ with $V = (U^{-1})^t$. We can show, by methods similar to those above, that $P^{-1}VP$ is unimodular. But then

$$(P^{-1}VP)^{-1} = P^{-1}V^{-1}P = P^{-1}U^tP = (PUP^{-1})^t$$

is unimodular, so PUP^{-1} is unimodular.

So ψ^h is a well-defined function from $C(N, p^hd')$ to $C(N, p^{3g-h}d')$. But then it is clear that ψ^{3g-h} provides an inverse for ψ^h. So each such function is a one-to-one correspondence and the proof of Theorem 4 is complete. □

There is a similar result for $p = 2$.

Theorem 5. *Let N and d be given by equations* (4) *and* (5). *Suppose that $2^g \parallel N$ and $2^h \parallel d$. Write d as $2^hd'$. Then there is a one-to-one correspondence ψ between $C(N, 2^hd')$ and $C(N, 2^{3g-h-2}d')$. This correspondence is defined by $\psi(\overline{f}) = \overline{f_2}$, where we may assume that f is as given below and then define f_2 accordingly:*

If $h = g-2$, then

$$f = \begin{pmatrix} 2^{g-2}a & b & c \\ r & 2^{g-1}s & 2^{g-1}t \end{pmatrix}, \qquad f_2 = \begin{pmatrix} a & 2^gb & 2^gc \\ 2^gr & 2^gs & 2^gt \end{pmatrix}.$$

If $g \leq h \leq 2g-2$, then

$$f = \begin{pmatrix} 2^{g-2}a & 2^{h-g}b & c \\ 2^{h-g+1}r & 2^{g-1}s & 2^{g-1}t \end{pmatrix}, \qquad f_2 = \begin{pmatrix} a & 2^{2g-h-2}b & 2^{g-2}c \\ 2^{g-1}r & 2^{g-1}s & 2^{2g-h-1}t \end{pmatrix}.$$

If $h = 2g$, *then*

$$f = \begin{pmatrix} 2^g a & 2^g b & c \\ 2^g r & 2^g s & 2^g t \end{pmatrix}, \qquad f_2 = \begin{pmatrix} a & b & 2^{g-2}c \\ 2^{g-1}r & 2^{g-1}s & t \end{pmatrix}.$$

The proof of Theorem 5 is similar to that of Theorem 4 (and Lemma 2), with the extra care which the prime 2 usually requires. For the computational purposes which are our focus, however, we can see that the effect of these correspondences may be more easily obtained by the use of the reciprocal correspondence of Theorem 3. So we will omit the proof of this theorem.

We combine and summarize the results of this section as:

Corollary 2. *Let* N *and* d *be given by equations* (4) *and* (5), *and let* $e = 2^{e_0}p_1^{e_1}\cdots p_k^{e_k}$. *Then there is a one-to-one correspondence between* $C(N, d)$ *and* $C(N, e)$ *if for all* $1 \le i \le k$, $e_i = d_i$ *or* $e_i = 3n_i - d_i$, *and if* $e_0 = d_0$ *or* $e_0 = 3n_0 - d_0 - 2$. *Thus, representatives of all classes of ternary forms of level* N *can be obtained by applying a sequence of the functions* ψ_p *to the classes in* $C(N, d)$ *with* $d_0 \le \frac{3}{2}n_0 - 1$, *and* $d_i \le \frac{3}{2}n_i$ *for* $1 \le i \le k$.

For example, consider $N = 60 = 2^2 \cdot 3 \cdot 5$. By Theorem 2, there are twelve potential discriminants for ternary forms of level 60. But we need only find the reduced forms for two of them: $d = 15$ and $d = 60$. Applying the map ψ_3 to the set $C(60, 15)$ gives us the entire set $C(60, 45)$. Applying ψ_5 to these two sets yields all of $C(60, 75)$ and $C(60, 225)$. Taking the reciprocals of all of these forms gives us all of the elements in $C(60, d)$ for $d =$ 3600, 1200, 720, and 240. (We could obtain the same sets by applying the map ψ_2 at this point.) Applying ψ_3 to $C(60, 60)$ gives us all of $C(60, 180)$. We may take reciprocals of those forms to obtain all forms in $C(60, 900)$ and $C(60, 300)$. Thus, we have representatives (but not in general the reduced forms) of all classes of ternary forms with level 60. Note that if $N/4$ is squarefree, then only two discriminant values, $d = N/4$ and $d = N$, need to be considered.

The process outlined here can be effectively computed for many values of N. Using this method, along with an algorithm for finding the reduced form in the class of a given positive definite ternary form, the author has found all reduced forms with level $N \le 1500$, and all with $N \le 4000$ for which $N/4$ is squarefree (349,186 forms in all). In Table 1 in the supplement to this issue, we present a small part of these results—the reduced forms with level $N \le 100$. Note that in that table, the forms are ordered so as to preserve the effect of the ψ-maps defined above. That is, suppose that $p^g \parallel N$ and that $d = p^h d'$ with $\gcd(p, d') = 1$. If the forms listed to the right of d in Table 1 are in order $f_1, \ldots, f_n$, then the forms listed to the right of $d_1 = p^{3g-h}d'$ ($p^{3g-h-2}d'$, if $p = 2$) are in order $\psi_p(f_1), \ldots, \psi_p(f_n)$.

3. Levels and genera

In this section, we return to the application mentioned in the introduction, that is, the relation between (ternary) quadratic forms and (weight 3/2) modular and cusp forms. We first consider another classification of quadratic forms. Two integral quadratic forms are said to be *semi-equivalent* if they are equivalent over the p-adic integers for all primes p, and are equivalent over the real numbers (see [3] or [6] for more details). Semi-equivalent forms are said to be in the same *genus* (pl. *genera*) of forms. Equivalent forms are semi-equivalent, so we may speak of a class of forms as belonging to a genus.

For ternary forms, semi-equivalence can be tested as follows. Let f be a ternary form and ϕ its reciprocal, as given in equations (1) and (2) above. Let m and μ be the divisors of these forms. We can assume, by Proposition 2, that a and γ are relatively prime to $m\mu$. If p is an odd prime and $p \mid m$, define a symbol (f/p) to be the Legendre symbol (a/p). Similarly, if $p \mid \mu$, define (ϕ/p) to be (γ/p). If $16 \mid m$, let $(f/4) = (-1)^{(a-1)/2}$. If $32 \mid m$, let $(f/8) = (-1)^{(a^2-1)/8}$. Define $(\phi/4)$ and $(\phi/8)$ analogously if μ is divisible by 16 or 32. We will refer to these symbols (whichever ones are defined) as the collection of *genus symbols* for f.

Proposition 4 [3, pp. 378–384; 4, pp. 51, 52]. *Let f and g be primitive, positive definite ternary forms. Then f and g are in the same genus if and only if they have the same discriminant and level* (*and thus the same values of m and μ*) *and the same collection of genus symbols.*

Remarks. The definition of genus symbols given here is adapted from the definition of characters in [4]. That genus symbols are well defined can be shown directly; the proof is omitted. Proposition 4 can be established by showing that two ternary forms have the same genus symbols if and only if they have the same p-adic symbols as defined in [3]. It can also be shown directly that semi-equivalent forms have the same discriminant and level. Note that by Proposition 4, it is obvious that the reciprocals of semi-equivalent forms are semi-equivalent. Similarly, one can show that the maps of Theorems 4 and 5 are genus-preserving, that is, if two classes of forms are in the same genus, then so are ψ_p applied to those classes, as defined.

We note also a result on the existence of forms having a particular collection of genus symbols.

Proposition 5 [6, Theorem 46]. *Let $m = 2^{m_0} p_1^{m_1} \cdots p_k^{m_k}$ and $\mu = 2^{\mu_0} p_1^{\mu_1} \cdots p_k^{\mu_k}$ be two integers subject to the conditions that $m_0 \neq 1$, $\mu_0 \neq 1$, and $m_0 + \mu_0 \geq 4$.* (*Each p_i is a distinct odd prime; we do not assume that m_i and μ_i are both positive.*) *Let h* (*respectively η*) *equal ± 1 as $m/2^{m_0}$* (*respectively $\mu/2^{\mu_0}$*) *is congruent to $\pm 1 \pmod 4$. For $i = 1, \ldots, k$, let the symbols (f/p_i) and (ϕ/p_i) be chosen independently as ± 1; similarly choose $(f/4)$, $(f/8)$, $(\phi/4)$, and $(\phi/8)$.*

Then there is a primitive, positive definite ternary form f with reciprocal ϕ so that f has discriminant $d = m^2\mu/16$, level $N = m\mu/4$, and genus symbols

$$\left(\frac{f}{p_i}\right) \text{ if } m_i > 0, \quad \left(\frac{f}{4}\right) \text{ if } m_0 \geq 4, \quad \left(\frac{f}{8}\right) \text{ if } m_0 \geq 5,$$

$$\left(\frac{\phi}{p_i}\right) \text{ if } \mu_i > 0, \quad \left(\frac{\phi}{4}\right) \text{ if } \mu_0 \geq 4, \quad \left(\frac{\phi}{8}\right) \text{ if } \mu_0 \geq 5,$$

if and only if the following conditions hold:

$$(6) \qquad \left(\frac{f}{8}\right)^{m_0} \left(\frac{\phi}{8}\right)^{\mu_0} \prod_{i=1}^{k} \left(\frac{f}{p_i}\right)^{m_i} \left(\frac{\phi}{p_i}\right)^{\mu_i} = (-1)^{((f/4)+\eta)((\phi/4)+h)/4} (-1)^{(h+1)(\eta+1)/4},$$

and

$$(7) \qquad \left(\frac{\phi}{4}\right) = -h \quad \text{if } m_0 = 0, \qquad \left(\frac{f}{4}\right) = -\eta \quad \text{if } \mu_0 = 0.$$

Remarks. The notation is again adapted from that of Dickson [4, pp. 51–54]. It can be shown directly, by methods similar to those of Lemma 1, that conditions (6) and (7) hold for any ternary form f. (Lemma 1 is in fact a special case of Proposition 5.) The existence of ternary forms subject to these conditions follows from Theorem 46 in [6].

If f is a positive definite ternary quadratic form of level N and discriminant d, then $\theta(f)$, as defined in the introduction, is in $M_{3/2}(N, \chi_d)$. Recall that χ_d, a Dirichlet character modulo N, depends only on the squarefree part of d. In keeping up the connection between modular forms and quadratic forms, we will say that a ternary form f has *character* q if $d = qr^2$ and q is squarefree. If f has level N, its character is a squarefree divisor of $N/4$.

If f_1 and f_2 are equivalent forms, then $\theta(f_1) = \theta(f_2)$. Let $c = c_q(N)$ be the number of classes of ternary forms having level N and character q. We thus have c forms in $M_{3/2}(N, \chi_q)$. These forms might not be linearly independent, but c provides an upper limit on the number of independent modular forms which arise directly from quadratic forms.

If f_1 and f_2 are semi-equivalent ternary forms of level N and character q, then $\theta(f_1) - \theta(f_2)$ is in $S_{3/2}(N, \chi_q)$. Let $g = g_q(N)$ be the number of genera of ternary forms with level N and character q. Of course, $g \le c$ in all cases. Suppose that a genus of forms contains n classes, say with $f_1, f_2, \dots, f_n$ as class representatives. Then $\theta(f_1) - \theta(f_2), \dots, \theta(f_1) - \theta(f_n)$ are cusp forms which might be independent. Any other difference, though, is easily seen to be a linear combination of these $n-1$ forms. Thus there is a maximum of $n-1$ linearly independent cusp forms arising directly from this genus.

Now suppose that the c classes of level N and character q are partitioned into the corresponding g genera so that the first genus contains c_1 classes, the second genus contains c_2 classes, and so on. Then the maximum number of linearly independent cusp forms which can be constructed from these classes is

$$(c_1 - 1) + (c_2 - 1) + \cdots + (c_g - 1) = \sum_{i=1}^{g} c_i - \sum_{i=1}^{g} 1 = c - g.$$

Let $s_q(N) = c_q(N) - g_q(N)$.

We can calculate $g_q(N)$ for all values of N and q. First let $g(N, d)$ denote the number of genera of forms of level N and discriminant d.

Lemma 3. *Let N and d be given by equations* (4) *and* (5). *For $1 \le i \le k$, let*

$$r_i = \begin{cases} 1 & \text{if } d_i = n_i \text{ or } d_i = 2n_i, \\ 2 & \text{if } n_i < d_i < 2n_i. \end{cases}$$

Let $r = r_1 + \cdots + r_k$. Then $g(N, d) = c \cdot 2^r$, where c is defined as follows:

(I) *If N and d are both squares, then*

$$c = \begin{cases} 1 & \text{if } 2n_0 - 4 \le d_0 \le n_0 + 2, \\ 4 & \text{if } n_0 + 4 \le d_0 \le 2n_0 - 6, \\ 2 & \text{otherwise.} \end{cases}$$

(II) *If N and d are not both squares, then*

$$c = \begin{cases} 1/2 & \text{if } n_0 = 2,\ d_0 = 0 \text{ or } 4, \\ 1 & \text{if } n_0 = 2,\ d_0 = 2;\ n_0 = 3,\ d_0 = 3 \text{ or } 4;\ n_0 = 4,\ d_0 = 5, \\ 3/2 & \text{if } n_0 = 4,\ d_0 = 4 \text{ or } 6, \\ 2 & \text{if } n_0 = 5,\ d_0 = 6 \text{ or } 7;\ n_0 = 6,\ d_0 = 8. \end{cases}$$

If n_0 and d_0 are not among these exceptional cases, then

$$c = \begin{cases} 1 & \text{if } d_0 = n_0 - 2 \text{ or } 2n_0, \\ 3 & \text{if } d_0 = n_0 \text{ or } 2n_0 - 2, \\ 4 & \text{if } d_0 = n_0 + 1,\ n_0 + 2,\ 2n_0 - 4, \text{ or } 2n_0 - 3, \\ 8 & \text{if } n_0 + 3 \le d_0 \le 2n_0 - 5. \end{cases}$$

Proof. We want to count the number of different collections of genus symbols which are allowed by conditions (6) and (7) of Proposition 5. Notice that in that statement, some symbols are defined, and may play a part in equation (6), which are not part of the collection of genus symbols. We will say that a symbol is "relevant" if it is in fact a genus symbol. For example, (f/p_i) is relevant if $n_i < d_i$, and (ϕ/p_i) is relevant if $d_i < 2n_i$. So r is the number of relevant symbols involving the odd primes. Call these the "odd" symbols. For the others, $(f/4)$ (resp. $(f/8)$) is relevant if $d_0 \ge n_0 + 2$ (resp. $n_0 + 3$); $(\phi/4)$ (resp. $(\phi/8)$) is relevant if $d_0 \le 2n_0 - 4$ (resp. $2n_0 - 5$).

Case I. If N and d are squares, then so are m and μ, so we have that $h = 1 = \eta$. Each m_i and μ_i is even, so equation (6) becomes

$$1 = -(-1)^{((f/4)+1)((\phi/4)+1)/4}.$$

By Lemma 1, neither m nor μ can be odd, so condition (7) does not apply in this case. Equation (6) reduces to requiring only that $(f/4) = 1 = (\phi/4)$. Otherwise, we see that the r odd symbols can be chosen independently, as can $(f/8)$ and $(\phi/8)$. So the number of possibilities for these choices is:

2^r if neither $(f/8)$ nor $(\phi/8)$ is relevant,
$2 \cdot 2^r$ if only one of $(f/8)$ and $(\phi/8)$ is relevant,
$4 \cdot 2^r$ if both $(f/8)$ and $(\phi/8)$ are relevant.

With the facts noted in the previous paragraph, and the fact that here n_0 and d_0 are both even, we get the result of the theorem.

Case II. Suppose that N and d are not both squares. We can rewrite equation (6) as

$$(6) \qquad \left(\frac{f}{8}\right)^{d_0 - n_0} \left(\frac{\phi}{8}\right)^{d_0} \prod_{i=1}^{k} \left(\frac{f}{p_i}\right)^{d_i - n_i} \left(\frac{\phi}{p_i}\right)^{d_i} = (-1)^{((f/4)+\eta)((\phi/4)+h)/4}(-1)^{(h+1)(\eta+1)/4}.$$

Note that if n_0 and d_0 are both even, then in this case we must have that n_i or d_i is odd for some $1 \le i \le k$. So then $r > 0$, and it makes sense to speak of choosing $r - 1$ odd symbols in a particular way. (In all other cases, we do not assume that $k > 0$.)

(1) Let $d_0 = n_0 - 2$. Here, $(\phi/4)$ is relevant, as is $(\phi/8)$ if $n_0 \geq 3$. Neither $(f/4)$ nor $(f/8)$ is relevant. In this case, m is odd, so we know that $(\phi/4) = -h$. Equation (6) becomes

$$\left(\frac{\phi}{8}\right)^{n_0} \prod_{i=1}^{k} \left(\frac{f}{p_i}\right)^{d_i - n_i} \left(\frac{\phi}{p_i}\right)^{d_i} = (-1)^{(h+1)(\eta+1)/4}.$$

If n_0 is odd, then we may choose the r odd symbols as we like; the value of $(\phi/8)$ is then determined by this equation. If n_0 is even, then the value of $(\phi/8)$ plays no part in equation (6). We can choose $(\phi/8)$ and $r-1$ of the odd symbols independently. The last odd symbol is then determined. So if $n_0 \geq 3$, we have r independent choices for the relevant symbols, for a total of 2^r possibilities. If $n_0 = 2$, then there are $2^{r-1} = \frac{1}{2} \cdot 2^r$ possibilities for the collection of genus symbols.

The case in which $d_0 = 2n_0$ is the same (with f and ϕ interchanged).

(2) Let $d_0 = n_0$. Here, $(\phi/4)$ is relevant if $n_0 \geq 4$, $(\phi/8)$ if $n_0 \geq 5$. Neither $(f/4)$ nor $(f/8)$ is relevant. Equation (6) becomes

$$\text{(6a)} \qquad \left(\frac{\phi}{8}\right)^{n_0} \prod_{i=1}^{k} \left(\frac{f}{p_i}\right)^{d_i - n_i} \left(\frac{\phi}{p_i}\right)^{d_i} = (-1)^{((f/4)+\eta)((\phi/4)+h)/4}(-1)^{(h+1)(\eta+1)/4}.$$

Suppose first that $n_0 \geq 4$. If $(\phi/4) = -h$, then the right-hand side of equation (6a) is $(-1)^{(h+1)(\eta+1)/4}$, which determines the left-hand side. As in subcase (1) above, we have r independent choices for $(\phi/8)$ and the odd symbols. (Again, which ones we can choose depends on the parity of n_0.) On the other hand, if $(\phi/4) = h$, then the right-hand side is $\pm(-1)^{(h+1)(\eta+1)/4}$ depending on the value of $(f/4)$. We have $r+1$ free choices for the odd symbols and for $(\phi/8)$. ($(f/4)$ is then determined but is not relevant.) So if $n_0 \geq 5$, then there is a total of $2^r + 2^{r+1} = 3 \cdot 2^r$ possibilities for the relevant symbols. If $n_0 = 4$, then $(\phi/8)$ is not relevant, so the total number of possibilities is $2^{r-1} + 2^r = \frac{3}{2} \cdot 2^r$.

Now if $n_0 < 4$, then neither $(f/4)$ nor $(\phi/4)$ is relevant. By choosing their values as we like, we have r free choices for the odd symbols. So there are 2^r possibilities if $n_0 = 3$ or $n_0 = 2$.

The case in which $d_0 = 2n_0 - 2$ is the same.

(3) Let $d_0 = n_0 + 1$. Here, $(\phi/4)$ is relevant if $n_0 \geq 5$, $(\phi/8)$ if $n_0 \geq 6$. Neither $(f/4)$ nor $(f/8)$ is relevant. Equation (6) becomes

$$\left(\frac{f}{8}\right)\left(\frac{\phi}{8}\right)^{n_0+1} \prod_{i=1}^{k} \left(\frac{f}{p_i}\right)^{d_i - n_i} \left(\frac{\phi}{p_i}\right)^{d_i} = (-1)^{((f/4)+\eta)((\phi/4)+h)/4}(-1)^{(h+1)(\eta+1)/4}.$$

By choosing $(f/8)$ as we like, we have free choices for each of the relevant symbols. So the total number of possibilities is $2^{r+2} = 4 \cdot 2^r$ if $n_0 \geq 6$, $2^{r+1} = 2 \cdot 2^r$ if $n_0 = 5$, and 2^r if $n_0 \leq 4$.

The case in which $d_0 = 2n_0 - 3$ is the same.

(4) Let $d_0 = n_0 + 2$. Now $(f/4)$ is relevant, but $(f/8)$ is not; $(\phi/4)$ is relevant if $n_0 \geq 6$, $(\phi/8)$ if $n_0 \geq 7$. Equation (6) becomes (6a) again. If

$n_0 \geq 6$, then choose $(f/4)$ and $(\phi/4)$ independently. The left-hand side of equation (6a) is then determined. There are r free choices for $(\phi/8)$ and the odd symbols. So if $n_0 \geq 7$, then there is a total of $2^{r+2} = 4 \cdot 2^r$ possibilities for the relevant symbols. If $n_0 = 6$, there are $2 \cdot 2^r$ total possibilities. The cases in which $n_0 \leq 5$ are already accounted for.

The case in which $d_0 = 2n_0 - 4$ is the same.

(5) Finally, let $n_0+3 \leq d_0 \leq 2n_0-5$. Here, $(f/4)$, $(f/8)$, $(\phi/4)$, and $(\phi/8)$ are all relevant. Choosing $(f/4)$ and $(\phi/4)$ determines the left-hand side of equation (6). We can choose $r+1$ of the remaining symbols freely. The total number of possible collections is $2^{r+3} = 8 \cdot 2^r$.

So the proof of Lemma 3 is complete. □

Notice that the number of genera of ternary forms of level N and discriminant d is positive in all cases listed. So the number of such classes must be positive as well. This proves the remark which concludes §1.

With N given by equation (4), let $q = 2^{q_0} p_1^{q_1} \cdots p_k^{q_k}$ with each q_i equal to 0 or 1, and $q_0 = 0$ if $n_0 = 2$. So q is a possible character for a ternary form of level N. We can now calculate $g_q(N)$ as

$$g_q(N) = \sum_d g(N, d) = \sum_d c \cdot 2^r,$$

where c and r are as given in Lemma 3, and the sum is taken over all d, given by equation (5), for which $\mathrm{sf}(d) = q$. A sum over all such d can be viewed as a sum over $(k+1)$-tuples $(d_0, d_1, \ldots, d_k)$ for which $d_i \equiv q_i \pmod 2$.

Let r_i be given as in the statement of Lemma 3. Then we have

$$g_q(N) = \sum_d c \cdot 2^r = \sum_{(d_0, \ldots, d_k)} c \cdot 2^{r_1} \cdots 2^{r_k}.$$

But c depends only on d_0 (once it is determined whether Case I or Case II applies), while 2^{r_i} depends only on d_i. So we can see that

$$g_q(N) = \sum_{d_0} c \cdot \sum_{d_1} 2^{r_1} \cdots \sum_{d_k} 2^{r_k}.$$

Theorem 6. *Let N be given by equation* (4) *and let q be a squarefree divisor of $N/4$. Let $g_q(N)$ be the number of genera of positive definite ternary forms having level N and character q. Then*

$$g_q(N) = C \prod_{i=1}^{k} (2n_i),$$

where C is a constant defined as follows:

(I) *If N is a square and $q = 1$, then*

$$C = \begin{cases} 1 & \text{if } n_0 = 2, \\ 2 & \text{if } n_0 = 4, \\ 5 & \text{if } n_0 = 6, \\ 2(n_0 - 4) & \text{if } n_0 \geq 8. \end{cases}$$

(II) *If N is not a square or $q \neq 1$, then*

$$C = \begin{cases} 2 & \text{if } n_0 = 2 \text{ or } 3, \\ 1 & \text{if } n_0 = 4 \text{ and } q \text{ is even}, \\ 5 & \text{if } n_0 = 4 \text{ and } q \text{ is odd}, \\ 6 & \text{if } n_0 = 5, \\ 10 & \text{if } n_0 = 6 \text{ and } q \text{ is odd}, \\ 4(n_0 - 4) & \text{if } n_0 = 6 \text{ and } q \text{ is even, or if } n_0 \geq 7. \end{cases}$$

Proof. In light of the remarks above, we need only show that

$$\sum_{d_i} 2^{r_i} = 2n_i \quad \text{for } i = 1, \dots, k, \quad \text{and} \quad \sum_{d_0} c = C.$$

For $i = 1, \dots, k$, we will consider three cases:

(i) If n_i is even and $q_i = 0$, then the possibilities for d_i are n_i, $n_i + 2, \dots, 2n_i - 2$, and $2n_i$. Then $r_i = 1$ for $d_i = n_i$ and $d_i = 2n_i$, and $r_i = 2$ in all other cases. Notice that $n_i + 2 \leq d_i \leq 2n_i - 2$ for $(n_i - 2)/2$ even values of d_i. Thus

$$\sum_{d_i} 2^{r_i} = 2^1 + \frac{n_i - 2}{2}(2^2) + 2^1 = 4 + 2(n_i - 2) = 2n_i.$$

(ii) If n_i is even and $q_i = 1$, then $d_i = n_i + 1$, $n_i + 3, \dots, 2n_i - 1$. There are $n_i/2$ such values of d_i, and $r_i = 2$ in each case, so

$$\sum_{d_i} 2^{r_i} = \frac{n_i}{2}(2^2) = 2n_i.$$

(iii) Suppose that n_i is odd. If $q_i = 0$, then $d_i = n_i + 1, \dots, 2n_i - 2$, $2n_i$. If $q_i = 1$, then $d_i = n_i$, $n_i + 2, \dots, 2n_i - 1$. In either case, there is one value of d_i for which $r_i = 1$ and $(n_i - 1)/2$ values for which $r_i = 2$. So

$$\sum_{d_i} 2^{r_i} = 2^1 + \frac{n_i - 1}{2}(2^2) = 2 + 2(n_i - 1) = 2n_i.$$

Now let c be defined as in Lemma 3.

Case I. Suppose that N is a square and that $q = 1$ (so that d is also a square). In particular, n_0 and d_0 are both even. If $n_0 \geq 8$, then there are the following possibilities for d_0: n_0, $n_0 + 2$, $2n_0 - 4$, $2n_0 - 2$, and for $(n_0 - 8)/2$ values, $n_0 + 4 \leq d_0 \leq 2n_0 - 6$. By Lemma 3, for the first four values, $c = 2$, and for the others, $c = 4$. So we have that

$$C = 2 + 2 + 2 + 2 + \frac{n_0 - 8}{2}(4) = 2(n_0 - 4).$$

For the other values of n_0, we have

$$\begin{aligned} n_0 = 6 &\Rightarrow d_0 = 6, 8, \text{ or } 10 \Rightarrow C = 2 + 1 + 2 = 5, \\ n_0 = 4 &\Rightarrow d_0 = 4 \text{ or } 6 \quad\quad \Rightarrow C = 1 + 1 = 2, \\ n_0 = 2 &\Rightarrow d_0 = 2 \quad\quad\quad\quad \Rightarrow C = 1. \end{aligned}$$

Case II. Suppose that N is not a square or that $q \neq 1$ (that is, N and d are not both squares). We consider three subcases here.

(i) Suppose that n_0 and d_0 are even, so that q is odd. If $n_0 \geq 8$, then $d_0 = n_0 - 2$, n_0, $n_0 + 2$, $2n_0 - 4$, $2n_0 - 2$, $2n_0$, and for $(n_0 - 8)/2$ values, $n_0 + 4 \leq d_0 \leq 2n_0 - 6$. Then by Lemma 3,

$$C = 1 + 3 + 4 + 4 + 3 + 1 + \frac{n_0 - 8}{2}(8) = 4(n_0 - 4).$$

Otherwise, we have

$$\begin{aligned} n_0 = 6 &\Rightarrow d_0 = 4, 6, 8, 10, \text{ or } 12 \Rightarrow C = 1 + 3 + 2 + 3 + 1 = 10, \\ n_0 = 4 &\Rightarrow d_0 = 2, 4, 6, \text{ or } 8 \qquad \Rightarrow C = 1 + \frac{3}{2} + \frac{3}{2} + 1 = 5, \\ n_0 = 2 &\Rightarrow d_0 = 0, 2, \text{ or } 4 \qquad \Rightarrow C = \frac{1}{2} + 1 + \frac{1}{2} = 2. \end{aligned}$$

(ii) Suppose that n_0 is even, and d_0 is odd, so that q is even. For $n_0 \geq 6$, we have that $d_0 = n_0 + 1$, $2n_0 - 3$, and for $(n_0 - 6)/2$ values, $n_0 + 3 \leq d_0 \leq 2n_0 - 5$. Then

$$C = 4 + 4 + \frac{n_0 - 6}{2}(8) = 4(n_0 - 4).$$

If $n_0 = 4$, then $d_0 = 5$ is the only possibility, so $C = 1$.

(iii) Suppose that n_0 is odd. We will assume that d_0 is even; the case in which d_0 is odd is similar. If $n_0 \geq 7$, then $d_0 = n_0 + 1$, $2n_0 - 4$, $2n_0 - 2$, $2n_0$, and for $(n_0 - 7)/2$ values, $n_0 + 3 \leq d_0 \leq 2n_0 - 6$. So then,

$$C = 4 + 4 + 3 + 1 + \frac{n_0 - 7}{2}(8) = 4(n_0 - 4).$$

Otherwise,

$$\begin{aligned} n_0 = 5 &\Rightarrow d_0 = 6, 8, \text{ or } 10 \Rightarrow C = 2 + 3 + 1 = 6, \\ n_0 = 3 &\Rightarrow d_0 = 4 \text{ or } 6 \qquad \Rightarrow C = 1 + 1 = 2. \end{aligned}$$

This completes the proof of Theorem 6. □

General results concerning the values of $c_q(N)$ and $s_q(N)$ are not apparent. However, combining the results of Theorems 4, 5, and 6, we can easily establish the following:

Theorem 7. *Let N be divisible by 4, and let $Q = \mathrm{sf}(N/4)$. Suppose that r is a squarefree divisor of $N/4$ and that $\gcd(r, Q) = 1$. Then*

$$c_r(N) = c_{rq}(N) \quad \text{and} \quad s_r(N) = s_{rq}(N)$$

if q is any divisor of Q.

In particular, if each prime in the unique factorization of $N/4$ appears with odd exponent, then the values $c_q(N)$ and $s_q(N)$ are independent of q.

Using the ternary forms listed in Table 1 (see Supplements section), the author has found bases for all spaces of cusp forms of weight $3/2$, level $N \leq 100$, and quadratic character. However, for larger values of N, the quadratic form method will not suffice in this direct way for construction of such bases. Suppose that $N = 4p$ for some prime p. (We restrict our attention to this case because then $S_{3/2}(N, \chi)$ does not contain any nontrivial subspaces of the form $S_{3/2}(M, \chi)$ with $M < N$.) It can be shown that

$$\dim S_{3/2}(N, \chi) = \begin{cases} (p-5)/4 & \text{if } p \equiv 1 \pmod 4, \\ (p-3)/4 & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

if $N = 4p$ with p an odd prime, and $\chi = \chi_1$ or $\chi = \chi_p$ [2, Theorem 2; 11, Theorem A].

In Table 2 in the supplement to this issue, we compare $\dim S_{3/2}(N, \chi_q)$ with our calculation of $s_q(N)$ for each $N \leq 4000$ with $N/4$ prime. It is apparent that, for these values, $s_q(N)$ does not increase as quickly as does $\dim S_{3/2}(N, \chi_q)$. Thus, the question of how useful the quadratic form method is in constructing a basis for a space of modular or cusp forms remains unresolved.

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