IK, Dec. 1995, Updated April/96

A speculation concerning regular ternary forms

Informal memo -- not for publication

1. Introduction. The speculation reported here arose a couple of weeks ago. Although it may be premature I decided to share it with people interested in regularity of ternary forms. If nothing else, it may be helpful as a framework for thinking about the problem If proved, the speculation would go a long way toward classifying regular odd ternary forms: one would start with Watson's list of 24 square-free ones in [12,13] and climb up till regularity expires.

Needless to say, at the time of writing the speculation is consistent with all the facts known to me. (April/96: the speculation holds up to discriminant 1000).

2. Terminology. Although case distinctions are ugly, I find it helpful to treat "odd" and "even" forms separately. In my opinion a shotgun wedding is not cost effective here. So: $ax^2 + by^2 + cz^2 + dxy + exz + fyz$ is even if d, e, f are all even and odd otherwise. As for discriminants, for even forms I use the determinant of the obvious attached matrix; for odd forms I follow Brandt and Intrau [10] by using half the determinant of the doubled up form. Thus the even form $x^2 + y^2 + z^2$ has discriminant 1 and the odd form $x^2 + y^2 + z^2 + xy + xz$ has discriminant 2.

For brevity, a "form" is always a positive definite ternary form.

3. Speculation. Suppose that D is the discriminant of a regular odd form and that p^2 divides D, p a prime. Then if p is odd I speculate that there exists a regular odd form of discriminant D/p; if p = 2 I speculate that there exists a regular odd form of discriminant D/4.

Remark. The stronger statement that one can pass to any divisor of D is too strong: there are regular odd forms of discriminants 52 and 78 but none of discriminant 26. However, exceptions seem to be few.

4. Census of odd regular forms. ~~Watson [12] announced that there are precisely 787 forms that are alone in their genera. So I take these as known. (Query: are details of this work of Watson available?) Appended to this note is a list of the 42 odd regular forms not alone in their genera that are known to me~~ plus a candidate. ~~Please send me additions, subtractions, or corrections.~~ According to Watson [14] there are 790 forms alone in their genera. See the postscript for more. The list below presents 43 regular forms not alone in their genera. The postscript has more.

5. Even forms. For even forms things are different: for instance, there is a regular even form of discriminant 52 but none of discriminant 13. Also: there is less information available. So I am not proposing a speculation at this time.

What regular forms are known? In addition to those alone in their genera let us set aside the 102 diagonal ones found by Jones in his thesis [5]; for details see [2, pp. 112-113] or [6]. To shorten the list still further let us also delete those forms that become diagonal when k(x^2 + xy + y^2) is replaced by k(x^2 + 3y^2). Then I only know two! One is the form x^2 + 2y^2 + 2yz + 13z^2 of discriminant 25, proved regular by Jagy [4]. The other is the form 5x^2 + 2xy + 5y^2 + 72z^2 of discriminant 1728, proved to be regular at the end of [6], by an impressive tour de force.

Update: I now know all the regular even forms up to discriminant 500. This includes 3 new ones not alone in their genera. See the final list. At the moment there are no even candidates.

Known regular odd forms not alone in their genera

Including
(~~[illegible] candidates~~)

Discriminant	Form	Reference
11	$x^2 + xy + 3y^2 + z^2$	Jones and Pall [6]
15	$x^2 + 2y^2 + yz + 2z^2$	Watson [13]
17	$x^2 + 2y^2 + 3z^2 + xy + 2yz$	Watson [13]
21	$x^2 + xy + 2y^2 + 3z^2$	Watson [13]
24	$x^2 + 3y^2 + 3z^2 + xy + 3yz$	IK [7]
27	$x^2 + xy + 7y^2 + z^2$	IK [8]. This form and the next one constitute a genus
27	$x^2 + 2y^2 + 4z^2 + xy + yz$	IK [8]
27	$x^2 + xy + y^2 + 9z^2$	Hsia [3]. This form and the next one constitute a genus. Note that both go to $x^2+3y^2+9z^2$ via the replacement $k(x^2+xy+y^2) \to k(x^2+3y^2)$
27	$x^2 + 3y^2 + 3yz + 3z^2$	Hsia [3]
32	$x^2 + 3y^2 + 3z^2 + xy + yz$	IK [7]
44	$x^2 + xy + 3y^2 + 4z^2$	~~This is a candidate. It sure is plausible: it represents all eligible numbers up to a million~~ Jagy
45	$x^2 + 3y^2 + 4z^2 + xz$	Schulze-Pillot [11]

48	$x^2+3y^2+5z^2+xz+3yz$	IK [7]
50	$x^2+2y^2+7z^2+xz+2yz$	Jagy. This is a corollary of the regularity of the even form $x^2+2y^2+2yz+13z^2$, mentioned in §5
56	$x^2+3y^2+5z^2+xz+yz$	IK [7]
63	$x^2+3y^2+3yz+6z^2$	Schulze-Pillot [11]
72	$x^2+3y^2+11z^2+xy+7yz$	Jagy [4]. In [7] this was one of four candidates
72	$x^2+3y^2+7z^2+xz+3yz$	IK [10].
75	$x^2+xy+4y^2+5z^2$	Schulze-Pillot [11]
80	$x^2+3y^2+7z^2+xz+yz$	IK [10]
81	$x^2+3y^2+7z^2+xz$	Hsia [3]
84	$x^2+5y^2+5z^2+xy+xz-2yz$	IK (not yet written up)
108	$x^2+4y^2+7z^2+xz$	IK [9]. This form and the next one constitute a genus
108	$x^2+5y^2+7z^2+xy+5yz$	IK [9]
108	$x^2+xy+y^2+36z^2$	Hsia [3]. Note that this goes to the even diagonal form $x^2+3y^2+36z^2$
108	$x^2+3y^2+10z^2+xz+3yz$	Hsia [3]
121	$x^2+xy+3y^2+11z^2$	IK (not yet written up)
120	$x^2+3y^2+11z^2+xz+3yz$	IK [10]

135	$x^2+3y^2+3yz+12z^2$	Schulze-Pillot [11]
135	$2x^2+xy+2y^2+9z^2$	Schulze-Pillot [11]
144	$x^2+3y^2+13z^2+xz+3yz$	IK [10]
147	$x^2+xy+2y^2+21z^2$	Schulze-Pillot [11]
216	$x^2+3y^2+19z^2+xz+3yz$	IK [10]
225	$2x^2+xy+2y^2+15z^2$	Schulze-Pillot [11]
243	$x^2+xy+7z^2+9z^2$	Hsia [3]
432	$x^2+3y^2+37z^2+xz+3yz$	Hsia et al [1]
289	$3x^2+5y^2+6z^2+3xy+2xz+yz$	Schulze-Pillot [11]
360	$x^2+3y^2+31z^2+xz+3yz$	IK [10]
441	$3x^2+3xy+6y^2+7z^2$	Schulze-Pillot [11]
675	$x^2+xy+4y^2+45z^2$	Schulze-Pillot [11]
675	$5x^2+6y^2+3yz+6z^2$	Schulze-Pillot [11]
972	$x^2+xy+7y^2+36z^2$	Hsia [3]
216	$3x^2+5y^2+5z^2+3xy+3xz+2yz$	IK. Not written up yet

Added April, 1996

~~240~~	$\cancel{x^2+5y^2+13z^2+xy+xz+2yz}$	
243	$2x^2+3y^2+11z^2+xz+3yz$	Jagy
392	$3x^2+3y^2+12z^2+xy+2xz-2yz$	IK
400	$3x^2+3y^2+12z^2+xy+2xz+2yz$	IK
432	$3x^2+9y^2+5z^2+3xz+3yz$	IK
484	$44x^2+y^2+3z^2+yz$	Jagy. This is a corollary of the form with disc. 44

600	$5x^2+7y^2+7z^2+5xy+5xz+6yz$	I K
648	$x^2+7y^2+25z^2+xy+xz+5yz$	I K

Candidates

240	$x^2+5y^2+13z^2+xy+xz+2yz$	
297	$x^2+6y^2+13z^2+xz+3yz$	
405	$2x^2+5y^2+11z^2+xy+2xz+2yz$	
720	$3x^2+5y^2+15z^2+3xy+3xz+3yz$	Regularity of this form is equivalent to regularity of the 240 form
1080	$3x^2+9y^2+11z^2+3xz+3yz$ ✓	
1125	$2x^2+7y^2+22z^2+xy+xz-6yz$	these forms lie in different genera
1125	$x^2+10y^2+29z^2+xz+5yz$	
1620	$5x^2+8y^2+11z^2+2xy+xz-4yz$	
2160	$5x^2+9y^2+15z^2+3xy+3xz+9yz$	
4536	$5x^2+9y^2+27z^2+3xy+3xz$	

Even regular forms not alone in their genera

		Jagy
324	$3x^2+4y^2+28z^2+4yz$	I K
400	$3x^2+3y^2+51z^2+2xy+2xz-2yz$	I K
480	$5x^2+8y^2+12z^2+4yz$	

References

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2. L. E. Dickson, Modern Elementary Theory of Numbers, Univ. of Chicago Press, 1939.

3. J. S. Hsia, Regular positive ternary quadratic forms, Mathematika 28(1981), 231-238.

4. W. C. Jagy, ~~More ternary quadratic forms, manuscript in preparation.~~ Five regular or nearly-regular ternary quadratic forms, submitted to Acta Arithmetica

5. B. W. Jones, Representation by positive ternary quadratic forms, Univ. of Chicago thesis, 1928.

6. B. W. Jones and G. Pall, Regular and semi-regular positive ternary quadratic forms, Acta Math. 70(1939), 165-191.

7. I. Kaplansky, Ternary positive quadratic forms that represent all odd positive integers, Acta Arithmetica 70(1995), 209-214.

8. _____, A second genus of regular ternary forms, ~~to appear~~ in Mathematika. 42(1995), 444-47

9. _____, A third genus of regular ternary forms, preprint, 6 pp. Submission for publication is not planned -- it ~~will be~~ is mentioned in [8] in a note added in proof.

10. _____, Three theorems on ternary forms, preprint, 7 pp. Submitted for publication and subsequently withdrawn. I plan to improve this and incorporate it in a later manuscript.

11. R. Schulze-Pillot, Darstellung durch definite ternäre quadratische Formen, J. of Number Theory 14(1982), 237-250.

12. G. L. Watson, One-class genera of positive ternary quadratic forms, Mathematika

19(1972), 96-104.

13. _____, Regular positive ternary quadratic forms, J. Lon. Math. Soc. 13(1976),.97-102.

14. — One-class genera of positive ternary quadratic forms, II Mathematika 22(1975), 1-11.

Postscript

Although the speculation remains a speculation, I thought it worth while to push through the project anyway. The help of Alexander Schiemann (Dortmund) was indispensable -- he sent me around 100 tables of forms and genera for certain discriminants. This was done in stages; as I climbed the tree branches dropped off. the largest discriminant was 31,752. The output for this one discriminant occupied 38 pages. There were 32 genera; the biggest one filled 6 pages. And in the end there were no regular forms of ~~the~~ discriminant 31,752.

In addition to the previous 43 forms, 17 new forms arose for a grand total of 60. Of these, 10 remain candidates (or 9 if one takes account of the fact that only one proof is needed for the discriminants 240 and 720.

Of course, the forms alone in their genera also showed up. The largest discriminant was 13,068. How does this compare with Watson [4]. Well, I found this paper tough going but I am slowly mastering it. As of now, I feel that the paper is sound but that he made errors (both ways) in identifying (without tables) forms alone in their genera. Forms ϕ_{38} and ϕ_{66} on page 3 have discriminant 6750, but there are no forms alone in their genera with that discriminant. And: 13,068 does not divide any ~~od~~ discriminant of an odd ϕ.