(1)

April, 1997

22 pages

Proofs of regularity

I. K.

There are 913 forms listed in [ ]. Of these, 794 are loners. That leaves 119. As of this writing, 22 are still candidates. That leaves 97. The purpose of this document is to account for these 97 proofs.

To shorten the list I shall omit the 20 diagonal regular nonloners that ~~ap~~ are part of this list. (The most convenient list of the 102 ~~ap~~ diagonal regular forms is on pp. 112–113 of [2]. Proofs of regularity are in [6].) I shall also omit the two "quasi-diagonal" even forms

108 : 1, 12, 12, 12, 0, 0 (this is also in Hsia [3])

1728 : 1, 48, 48, 48, 0, 0.

They become diagonal on replacing x^2+xy+y^2 by x^2+3y^2 and so need no further proof of regularity.

We are down to 75 and the list can begin.

(2)

Odd

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^
1 ; 11 : 1 1 3 0 1 0 Jones and Pall [6]
: 1 1 4 1 1 1

^
2 ; 15 : 1 2 2 1 0 0 Watson [17]
: 1 1 5 0 0 1

^
3 ; 17 : 1 2 3 2 1 1 Watson [17]
: 1 1 6 1 1 1

^
4 ; 21 : 1 2 3 0 0 1 Watson [17]
: 1 1 7 0 0 1

^
5 ; 24 : 1 3 3 3 1 1 IK [9] "ALL ODD" ✓
: 1 1 8 0 0 1

^
6 ; 27 : 1 1 7 0 1 0 }
7 ; 27 : 1 2 4 1 0 1 } IK [8] ✓

^
8 ; 27 : 1 1 9 0 0 1 }
9 ; 27 : 1 3 3 3 0 0 } Hsia [3] Note: these are "quasi-diagonal"

^
10 ; 32 : 1 3 3 1 0 1 IK [9] "ALL ODD" ✓
: 1 1 11 1 1 1

^
11 ; 44 : 1 3 4 0 0 1 Jagy [5]
: 1 1 15 1 1 1

^
12 ; 45 : 1 3 4 0 1 0 Schulze-Pillot [15]
: 1 1 15 0 0 1

^
13 ; 48 : 1 3 5 3 1 0 IK [12] "Three Theorems" ✓
: 1 1 16 0 0 1

^
14 ; 50 : 1 2 7 2 1 0 See Appendix I
: 1 1 17 1 1 1

^
15 ; 56 : 1 3 5 1 1 0 IK [9] "ALL ODD" ✓
: 1 1 19 1 1 1

^
16 ; 63 : 1 3 6 3 0 0 Schulze-Pillot [15]
: 1 1 21 0 0 1

^
17 ; 72 : 1 3 7 2 1 1 Jagy [4]
: 3 3 3 1 2 3

^
18 ; 72 : 1 3 7 3 1 0 — IK [12] ✓
: 1 1 24 0 0 1

^
19 ; 75 : 1 4 5 0 0 1 — Schulze-Pillot [15]
: 1 5 5 5 0 0

^
20 ; 80 : 1 3 7 1 1 0 — IK [12] 3 Thms ✓
: 1 1 27 1 1 1

^
21 ; 81 : 1 3 7 0 1 0 — Hsia [3]
: 1 1 27 0 0 1
: 3 3 4 3 3 3

^
22 ; 108 : 1 1 36 0 0 1 — Hsia [3]. Note: this is "quasi-diagonal".
: 3 3 4 0 0 3

^
23 ; 108 : 1 3 10 3 1 0 — Hsia [3]
: 3 4 4 4 3 3

^
24 ; 108 : 1 4 7 0 1 0 } IK [11] "THIRD GENUS" ✓
25 ; 108 : 1 5 7 5 1 1

^
26 ; 120 : 1 3 11 3 1 0 — IK [12] "3 Thms"
: 1 1 40 0 0 1

^
27 ; 121 : 1 3 11 0 0 1 — See Appendix II
: 3 4 4 -3 2 2

^
28 ; 135 : 1 3 12 3 0 0 — Schulze-Pillot [15]
: 1 1 45 0 0 1

^
29 ; 135 : 2 2 9 0 0 1 — Schulze-Pillot [15]
: 3 3 5 0 0 3

^
30 ; 144 : 1 3 13 3 1 0 — IK [12] "3 Thms"
: 1 1 48 0 0 1

^
31 ; 147 : 1 2 21 0 0 1 — Schulze-Pillot [15]
: 1 7 7 7 0 0

^
32 ; 189 : 2 3 8 0 1 0 — Jagy [5]
: 3 3 8 3 3 3

^
33 ; 216 : 1 3 19 3 1 0 — IK [12] "3 Thms"
: 1 1 72 0 0 1

34	216	3 5 5 2 3 3	See Appendix III
		3 3 8 0 0 3	
35	225	2 2 15 0 0 1	Schulze-Pillot [15]
		3 5 5 5 0 0	
	~~240~~	~~1 5 13 2 1 1~~	¡NO PROOF!
		~~4 5 5 5 2 4~~	
36	243	1 7 9 0 0 1	Hsia [3]
		1 9 9 9 0 0	
		4 4 4 -1 1 1	
37	243	2 3 11 3 1 0	Jagy [5]
		2 2 17 2 2 1	
		3 5 5 2 0 3	
38	289	3 5 6 1 2 3	Schulze-Pillot [15]
		3 6 6 -5 2 2	
	~~297~~	~~1 6 13 3 1 0~~	¡NO PROOF!
		~~4 4 6 -3 3 1~~	
39	360	1 3 31 3 1 0	IK [12] 3 THMS
		1 1 120 0 0 1	
40	392	3 3 12 -2 2 1	See Appendix III ✓
		5 5 5 3 3 3	
41	400	3 3 12 2 2 1	See Appendix III ✓
		5 5 7 5 5 5	
	~~405~~	~~2 5 11 2 2 1~~	¡NO PROOF!
		~~5 5 6 3 3 5~~	
42	432	1 3 37 3 1 0	~~Hsia~~ Benham, Earnest, Hsia, and Hung [1]
		1 1 144 0 0 1	non-spinor proof: IK, October 1999
		3 3 16 0 0 3	
		3 7 7 5 3 3	
43	432	3 5 9 3 0 3	See Appendix III ✓
		3 3 17 3 3 3	
44	441	3 6 7 0 0 3	Schulze-Pillot [15]
		3 7 7 7 0 0	

$\hat{45}$; 484 :	1 3 44 0 0 1		See Appendix II
:	5 5 5 -1 1 1		
$\hat{46}$; 600 :	5 7 7 6 5 5		See Appendix III
:	5 5 8 0 0 5		
$\hat{47}$; 648 :	1 7 25 5 1 1		See Appendix III
:	4 7 7 5 2 2		
$\hat{48}$; 675 :	1 4 45 0 0 1		Schulze-Pillot [15]
:	1 15 15 15 0 0		
$\hat{49}$; 675 :	5 6 6 3 0 0		Schulze-Pillot [15]
:	5 5 9 0 0 5		
~~; 720 :~~	~~3 5 15 3 3 3~~		¡NO PROOF!
~~:~~	~~5 5 11 4 5 5~~		
$\hat{0}$; 972 :	1 7 36 0 0 1		Hsia [3]
:	4 9 9 9 0 0		
:	7 7 7 5 5 5		
$\hat{51}$; 1080 :	3 9 11 3 3 0		Jagy [5]
:	3 3 41 3 3 3		
~~; 1125 :~~	~~1 10 29 5 1 0~~		¡NO PROOF!
~~:~~	~~4 9 11 9 4 3~~		
; 1125 :	~~2 7 22 -6 1 1~~		¡NO PROOF!
~~:~~	~~7 7 8 2 7 4~~		
$\hat{52}$; 1323 :	2 8 21 0 0 1		Jagy [5]
:	8 8 8 -5 5 5		
~~1620 :~~	~~5 8 11 4 1 2~~		¡NO PROOF!
~~:~~	~~5 5 23 4 5 5~~		
$\hat{53}$; 1800 :	5 11 11 7 5 5		Jagy [5]
:	5 5 24 0 0 5		
~~; 2160 :~~	~~5 9 15 9 3 3~~		¡NO PROOF!
~~:~~	~~5 5 29 1 2 5~~		
~~:~~	~~5 11 11 7 1 1~~		
2592 :	5 ~~9 17 6~~ 5 3		¡NO PROOF!

```
            :  5 5 27 3 3 1          ¡NO PROOF!   (2592)
            :  9 9 11 3 3 9
^
  ; 3375 :  2 15 32 15 1 0          ¡NO PROOF!
         :  8 8 17 8 8 1
^
  ; 4500 :  7  8 23  6  7  2        ¡NO PROOF!
         :  7 13 13  1  3  3
^
  ; 4536 :  5 9 27 0 3 3            ¡NO PROOF!
         :  9  9 20  6  6  9
         :  9 11 17  8  9  9
^
4 60; 5400 :  7  7 28 -2  2  1     Jagy [5]
54       : 13 13 13 11 11 11
^
    8232 :  5 13 33 -6  3  1        NO PROOF
         : 13 13 17 10 11  9
^
  ; 10125 :  9 11 29 -4  3  6       NO PROOF
          :  9 11 35 10  0  9
^
  ; 24696 :  11 15 39 -3  6  3      NO PROOF
          :  11 23 29 13 11  5
^
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Please check discriminants 8640, 14400, 43200 13 August 1996 (7)

Even

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^

; 16 : 1 1 16 0 0 0 Diagonal
: 2 2 5 2 2 0

^

55 4 ; 25 : 1 2 13 2 0 0 Jaggy [4]
: 2 2 9 2 2 2

^

; 64 : 1 2 32 0 0 0 Diagonal
: 2 4 9 4 0 0

^

; 64 : 1 4 16 0 0 0 Diagonal
: 4 4 5 0 4 0

^

56 8 ; 64 : 1 5 13 2 0 0 Jaggy [5]
: 4 5 5 4 0 4

^

; 108 : 1 12 12 12 0 0 QUASI-DIAGONAL
: 4 4 9 0 0 4

^

; 108 : 1 3 36 0 0 0 Diagonal
: 3 4 9 0 0 0

^

57 9 ; 108 : 1 4 28 4 0 0 See Appendix III
: 4 5 8 4 4 4 #1, near miss!

^

; 144 : 1 4 36 0 0 0 Diagona
: 4 4 9 0 0 0

^

; 192 : 1 8 24 0 0 0 Diagonal
: 4 8 9 8 4 0

^

; 224 : 3 6 14 4 2 2 ¡NO PROOF!
: 6 7 7 2 2 6

^

; 256 : 1 16 16 0 0 0 Diagonal
: 4 9 9 2 4 4

^

; 256 : 1 8 32 0 0 0 Diagonal
: 4 8 9 0 4 0

^
58 [illegible] ; 256 : 3 3 32 0 0 2 Jagy [5]
: 4 8 11 8 4 0

^
59 [illegible] ; 324 : 3 4 28 4 0 0 See Appendix III ✓
: 4 4 27 0 0 4
: 7 7 7 2 2 2

^
60 [illegible] ; 400 : 3 3 51 -2 2 2 See Appendix III ✓
: 8 8 11 8 8 8

^
; 432 : 1 12 36 0 0 0 Diagonal
: 4 9 12 0 0 0

^
61 [illegible] ; 448 : 5 8 12 0 4 0 See Appendix III ✓
: 5 5 20 4 4 2

^
; 512 : 1 8 64 0 0 0 Diagonal
: 4 8 17 0 4 0

^
; 576 : 3 8 24 0 0 0 Diagonal
: 8 11 11 10 8 8

^
; 768 : 1 16 48 0 0 0 Diagonal
: 4 16 17 16 4 0

^
; 960 : 5 8 24 0 0 0 Diagonal
: 8 13 13 6 8 8

^
; 1008 : 7 8 20 0 4 4 ¡NO PROOF!
: 7 7 27 6 6 6

^
62 [illegible] ; 1024 : 3 11 32 0 0 2 Jagy [5]
: 11 11 12 -4 4 10

^
63 [illegible] ; 1280 : 7 12 16 0 0 4 Jagy [5]

: 7 7 28 -4 4 2

^

64 23 ; 1296 : 5 8 36 0 0 4 Jagy [5]

: 8 9 20 0 8 0

^

; 1728 : 1 24 72 0 0 0 Diagonal

: 4 24 25 24 4 0

^

3 ; 1728 : 1 48 48 48 0 0 Quasi-diagonal

: 9 16 16 16 0 0

^

65 ; 1728 : 5 5 72 0 0 2 Jones and Pall [6]

: 8 12 21 12 0 0

^

) ; 1728 : 8 9 24 0 0 0 Diagonal

: 8 17 17 10 8 8

^

66 21 ; 1728 : 1 16 112 16 0 0 Jagy [5]

: 9 16 17 16 6 0

^

67 22 ; 1728 : 4 13 37 2 4 4 Jagy [5]

: 13 13 16 -8 8 10

^

; 2112 : 7 15 23 -6 2 6 ¡NO PROOF!

: 7 16 23 16 2 0

^

; 2304 : 3 16 48 0 0 0 Diagonal

: 12 16 19 16 12 0

^

; 2880 : 11 16 19 8 2 8 ¡NO PROOF!

: 11 11 27 6 6 6

^

5 ; 2880 : 8 15 24 0 0 0 Diagonal

: 8 23 23 22 8 8

^

68 27 ; 3136 : 3 19 56 0 0 2 Jagy [5]

: 12 19 19 -18 4 4

^

~~; 4800 : 1 40 120 0 0 0~~ Diagonal
~~: 4 40 41 40 4 0~~

^
69 ; 5184 : 3 16 112 16 0 0 Jagy [5]
: 16 16 27 0 0 16
: 19 19 19 -10 10 10

^
70 ; 5184 : 7 15 55 -6 2 6 Jagy [5]
: 15 16 28 16 12 0

^
~~; 6336 : 5 20 68 -8 4 4~~ ¡NO PROOF!
~~: 20 20 21 -12 12 8~~

^
71 ; 6400 : 3 27 80 0 0 2 Jagy [5]
: 12 27 27 -26 4 4

^
~~; 6912 : 1 48 144 0 0 0~~ Diagonal
~~: 4 48 49 48 4 0~~
~~: 9 16 48 0 0 0~~
~~: 16 25 25 14 16 16~~

^
72 ; 6912 : 9 17 48 0 0 6 Jagy [5]
: 17 17 32 -8 8 14

^
73 ; 6912 : 5 20 77 20 2 4 Jagy [5]
: 20 20 21 12 12 8

^
~~; 8000 : 11 16 51 8 2 8~~ ¡NO PROOF!
~~: 16 19 35 10 0 16~~

^
74 ; 8640 : 13 24 28 0 4 0 Jagy [5]
: 13 13 52 4 4 2

^
~~; 14400 : 3 40 120 0 0 0~~ Diagonal
~~: 12 40 43 40 12 0~~

^
~~; 14400 : 7 23 92 12 4 2~~ ¡NO PROOF!
~~: 23 28 28 -24 4 4~~

```
   ^
75 , ; 43200 :    9 41 120  0  0  6     Jaggy [5]
             :   36 41 41 -38 12 12
   ^
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Appendix I

Appendices 12-16 (5 pages)

Let f be an even form with odd discriminant. then there is a natural map, one-to-one and onto to g, an odd form with twice-odd discriminant. f and g enjoy many pleasant properties. One is that regularity of f implies regularity of g. By applying this to form #55 (even 25, proved regular by Jagy) one gets the regularity of form # 14 (~~even~~ ODD 50).

Remark 1. It is also true that regularity of g implies regularity of f but I have no a priori proof of this; it comes out in the wash after the classification is complete.

Remark 2. But # 55 is the only regular even nonloner with odd discriminant, anyway. Likewise #14 is the only odd nonloner with twice odd discriminant.

Appendix II

Let f be a form (odd or even) with a discriminant D divisible by the odd prime p to the first power. Suppose that p is "bad" for f. Then there is a natural map $f \to g$ to a certain form with discriminant pD. f and g are closely ~~relation~~ related: f (resp. g) represents A if and only if g (resp. f) represents pA. It follows that f is regular if and only if g is regular. These facts are in Watson [16], though not quite explicitly. The ideas also appear in [14]. Anyhow, they are easy to prove.

Normally one would expect to use this to "go up", and this way one gets the regularity of #27 (odd 121) from the regular form odd 11, and #45 (odd 484) from #11 (odd 44).

Remark 1. The argument can be used to give fresh proofs in a number of other cases. I skip the details

Remark 2. The method may be helpful some day in cracking some of the 22 candidates. One sees that ~~the regularity of odd~~ 240 ~~is equivalent to that of odd 720.~~ for the following four pairs, a proof of regularity for one forms yields regularity of the other

odd 240 $\longleftrightarrow$ odd 720

odd 8232 $\longleftrightarrow$ odd 24696

even 2112 $\longleftrightarrow$ even 6336

even 2880 $\longleftrightarrow$ even 14,400

Also: regularity of odd 2160 $\rightarrow$ regularity of 720

Remark 3. These ideas quickly lead to a proof that there is a natural one-to-one correspondence between odd forms of discriminant p and those of discriminant p^2. (p an odd prime)

APRIL 1997

Appendix III

As the work on the classification proceeded I found regularity proofs for ten forms:

#'s 34, 40, 41, 43, 46, (47, (57, 59, 60, 61.)

I jotted down the proofs in a big yellow pad in which I was collecting all sorts of remarks on ternary forms.

Later it turned out that ~~of these~~ the ~~forms (#'s 34, 40, 41, 43, 46, 47~~ first ~~six~~ five (~~i.e. the~~ all of them odd ~~forms~~) ~~all~~ yielded to the ~~simplified~~ simple unified method of [13].

There remain #47 (odd 648) and the last four (i.e. the even ones).

I have copied the relevant pages and placed them at the end of this document.

Remark 1. The extent to which these pages are legible and/or ~~problematic~~ understandable may be problematic. I plan to polish them some day and type them up.

Remark 2. I also proved that the mates of #'s 47 and 57 represent all eligible integers ≠ 1. These pages are included. Call these 'near misses.'

It may be of interest that there are some further such near misses among the mates of regulars. Two were proved by Jones and Pall [6], and ~~addition there are candidates~~ one by Jagy [4]. In addition there are candidates: the mate 4, 9, 11, 9, 4, 3 of the candidate 1, 10, 29, 5, 10 (odd 1125) & and the mates of #56, 70, 73 all appear to represent all eligible integers except 1.

1. J. W. Benham, A. G. Earnest, J. S. Hsia, and D. C. Hung, Spinor regular positive ternary quadratic forms, J. Lon. Math. Soc. 42 (1990), 1– 10

2. Dickson, Modern Elementary Theory of Numbers, U. of Chicago Press, 1939

3 Asia, Regular positive ternary forms, Mathematika 28 (1981), 231–238

~~Dickson, Modern Elementary Theory of Numbers~~

4. Jagy, Acta Arithmetica

5. Jagy, A collection of 22 proofs of regularity

6. Jones and Pall

7. Jagy, IK, and Schiemann There are 913 regular ternary forms

8. IK, A second genus of regular ternary forms.

9. —, Ternary positive quadratic forms

10. that represent all odd integers, Acta Arithmetica 73(1995), 209-214

11. —, A third genus of regular ternary forms

12. —, Three theorems on ternary forms

13. —, A method for possibly proving domination of forms. Application: a unified proof of regularity for 36 odd forms

14. —, Notes on the classification of regular ternary forms

15. Schulze-Pillot, Darstellung durch definite ternäre quadratische Formen, J. of Number theory 14 (1982), 237-250

16. Watson, thesis

17. Watson

page Jary [5] : to be typed! (22)

(2) ODD : 44
(3) 189
(4) 243
(5) 1080
1323
1800
(6) 5400

7

EVEN
(7) 64 — 1

256
(8) 1024 — 3
1280

(9) 1296
1728 A
1728 B — 4
3136

(10) 5184 A
5184 B
6400 — 6
6912 A
6912 B
8640

(11) 43,200 — 1

15

22