Received 23 June 1993

Dear Professor Kaplansky,

Again, I apologize for the delay in responding to your May 10th letter. I hope you got my e-message sent recently to "Nancy" of MSRI. Let me try to answer your questions in your letter and will refer to the numbers enumerated in that letter.

Re 1: No, I don't know any other "near misses" beside the two published ones in Jones/Pall.

Re 3: The Ramanujan form $R := X^2+Y^2+10Z^2$ represents all odd numbers of the form $2+3n$. Let me sketch the argument. Use prime $p=3$. Call the other lattice ("genus mate") $T := \langle 2,2,3,0,1,0\rangle$. Consider the graph G whose vertices are all lattices L in genus of R such that $L_p = R_p$ for all $p=3$. Two vertices L & K are "neighbors" if $[L : L\cap K]$ $(= [K : L\cap K]) = 3$. Each vertex has $p+1 = 4$ neighbors. {The neighbors are in a one-one correspondence with the isotropic lines of L/pL. } Let the matrix of the lattice T given above be with respect to basis $\{e_1, e_2, e_3\}$. If c is an exceptional integer for R then we want to show that $c \not\equiv 2 \pmod 3$. If not, then such a $c \equiv 2 \pmod 3$ must be represented by a vector $x \in T$, i.e. $Q(x) = c$, then the subgraph $G(T, x : 3)$ consisting of those vertices which contain x is a line through the vertex $T^{(0)} = T$. { This is because the orthogonal complement of the line $\mathcal{Q}_p x$ is a hyperbolic plane so that exactly two of the four neighbors of T contain x and each such neighbor again has two neighbors containing x, etc. In fact, locally at $p=3$ the localized lattices that contain x are just the following: $T^{(n)} = \langle 3^n e, 3^{-n} f\rangle \perp \langle x\rangle, n \in \mathcal{Z}$ if the unique isotropic lines of $x^\perp$ are given by $\{e, f\}$ with $B(e,f) = 1$. } Globally, the vertices of $G(T, x : 3)$ are given by $T^{(n)}$ with basis $\{e_1(n), e_2(n), e_3(n)\}$ and they are related by

$$e_1(n) = e_1(n+1)/3 + 2e_3(n+1)/3 = e_1(n-1) - 2e_3(n-1)/3$$

$$e_2(n) = e_2(n+1) = e_2(n-1)$$

$$e_3(n) = -e_1(n+1) + e_3(n+1) = e_1(n-1) + e_3(n-1)/3$$

If we write $x := a_1(n)e_1(n) + a_2(n)e_2(n) + a_3(n)e_3(n) \in T^{(n)},\ a_i(n) \in \mathcal{Z}$ then it is easy to see that $3(a_1(n+1)+a_1(n-1)) = 4a_1(n)$ for all n. Hence, if $a_1(n)$ is divisible by 3^s for some $s \geq 0$ and all n, then $a_1(n)$ is divisible by 3^{s+1} for all n. But, $a_1(n)$ is clearly divisible by 3^0, so divisible by 3^s for all $s \geq 0$, i.e. $a_1(n) = 0$ for all n. Similarly $a_3(n) = 0$ for all n. Now, $x := a_2(n)e_2(n)$ is the same for all n and $c = Q(x) = 2a_2(n)^2$ which is visibly represented by R contradicting hypothesis.

Re 7: Recently Conway was in Columbus for the "Monster Conference" at Ohio State. He also asked about your question about whether theta series of pos. definite quad. forms classifies the forms. [He just wanted to know about pos. ternaries.] Answer is as follows: it has been known for some time (since Witt in 1941, Abh. Hamburg?) that in dim 16 the two even unimodular not only have same ordinary (elliptic) theta series, but also have the same higher degree (Siegel modular forms) theta series up to degree $d \leq 3$ but at $d = 4$ they differ. For dim 12, Kneser had examples, det 4, in his 1961 paper dedicated to Siegel's 60th birthday in Math. Z.; for dim 8 Kitaoka had example in det 81, I believe, see his Proc. Japan Acad paper 1971. Recently Schiemann–apupil of Grunewald at Bonn, Arch Math 1991? had a short article on counter-example for dim 4 and det = 1729 (Ramanujan's taxicab number!) Conway/Sloane has a family of similar counter-examples in 1992 somewhere. It has been sort of expected that at dim 3, the theta series does classify. In fact, for dim ≤ 4 same theta series imply in same genus, but false for dim = 5. Supposedly, Schiemann also finished a computer assisted proof for this dim 3 statement. he also used computers to generate counter-example at dim 4 mentioned above.

Re 8: Thanks for mentioning to me Schulze-Pillot's letter saying his/Duke's result is not effective.

Re 10: You are right in observing that the defn of "semi-regular" was never explicitly defined in Jones/Pall, although it is sort of clear that they meant the "near miss" in your sense.

If time permits this summer I may try to look at some of your remaining forms. But, you are definitely right in saying it is very humbling indeed that arithmetic representations of simple looking pos. ternaries are still far from being fully understood—even when the class number is small. A form such as Ramanujan there is still no final answer to the list of "exceptions." On the other hand, I am so behind in a number of projects (some joint with others) that I must also try to finish some of them this summer.

Two final remarks:

1) Re 1, I think quaternary universal forms were handled by a former student of Arnold Ross, Margaret Willerding (1948), see referenc in Cassel book "Rational Quadratic Forms."

2) Conway in his recent visit mentioned the form $x^2 + 2y^2 + 4z^2 + yz$ (class number 3) having the interesting property, it represents all the pos. integers up to 30, but misses 31, and every other pos. ternary integral quadratic form misses some number ≤ 30. He didn't give me a proof. But I realized that my elementary proof that there is no ternary pos. universal quad. form shoulsd also work. Indeed, a few days later I was able to give a proof. The proof is sort of the expected type as you also use in your project...namely, representing small numbers, will quickly bound the determinant, which fortunately is rather small and then just rule them all out one by one, mostly by inspection almost. Nevertheless, it is a rather curious form.

I hope these remarks answer your explicit questions in your May 10th letter.

With best wishes,

John S. Hsia

P. S. As I type this letter, I hear memo from Isaac Newton Institute that Andrew Wiles has announced that he has proven Fermat. While he has not proved the full strength of the Shimura-Taniyama-Weil conjecture, he proved enough of it (semi-stable ell. curves are modular) which together with earlier results of Serre, Ribet, and Frey finally makes Fermat's Last Theorem a theorm! This is truly remarkable.