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Dear Dennis and John: (2 pages)

OK, so I didn't put ternaries aside after all. An idea just popped and it worked; so I couldn't resist rushing a letter to the two guys who might be interested.

Theorem. For any t, $f = x^2 + 3y^2 + 2yz + tz^2$ and $g = x^2 + y^2 + tz^2 + xy + xz$ represent the same numbers.

Remarks. 1. This holds for any t, positive, zero, or negative. So we have a nugget of information about indefinite forms.

2. For $t = 4$, these are the forms that Jones and Pall said represented the same numbers, without proof. A little while ago I was only able to prove inclusion one way, and was rescued by Dennis.

3. For $t = 9$, these are two of my "near misses" (they seem to represent all odd numbers with exactly one exception). So, although I have still not proved anything about them, at least I know that they will stand or fall together.

4. Of course if one form is regular, so is the other. But this seems to be no big deal. For $t = 1, 2, 3, 5$ all forms are alone in their genera. Otherwise, my guess is that all the others are non-regular.

5. The substitution of $x^2 + 3y^2$ for $x^2 + xy + y^2$ generates a horde of odd-even pairs that represent the same numbers. In addition, I know a small number of others.

The proof is nothing much -- just more of what I have been doing all along. I diagonalize.

Theorem$'$. The following statements are equivalent:

(a) f represents A.

(b) g represents A.

(c) $x^2 + 3y^2 + (3t - 1)z^2$ represents $3A$.

Proof. (a) $\Rightarrow$ (c). We have

(1) $$3f = 3x^2 + (3y + z)^2 + (3t - 1)z^2.$$

(b) $\Rightarrow$ (c). We have

(2) $$12g = 3(2y + x)^2 + (3x + 2z)^2 + (3t - 1)(2z)^2.$$

Thus $u^2 + 3v^2 + (3t - 1)w^2 = 12A$ with w even. So $u^2 + 3v^2$ is divisible by 4. One knows that $u^2 + 3v^2$ can be written $4(p^2 + 3q^2)$. Divide by 4 to get $p^2 + 3q^2 + (3t - 1)w^2 = 3A$.

(c) $\Rightarrow$ (a). We have $u^2 + 3v^2 + (3t - 1)w^2 = 3A$. We note $u^2 \equiv w^2 \pmod 3$; by a change of sign, if necessary, we arrange $u \equiv w \pmod 3$. Set $x = v$, $y = (u - w)/3$, $z = w$. One checks that $f(x, y, z) = A$.

(c) $\Rightarrow$ (b). This starts the same way. Then we set $x = 2(u - w)/3$, $y = v - (u - w)/3$, $z = w$ and find $g(x, y, z) = A$. (These equations for x, y, z, here and in (c) $\Rightarrow$(a), were of course obtained by solving (1) and (2).)

Regards

Kap