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DISCRIMINANT BOUNDS FOR SPINOR REGULAR TERNARY QUADRATIC LATTICES

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Dedicated to Professor J. S. Hsia on the occasion of his sixty-fifth birthday

Abstract

The main goals of the paper are to establish a priori bounds for the prime power divisors of the discriminants of spinor regular positive definite primitive integral ternary quadratic lattices, and to describe a procedure for determining all such lattices.

1. *Introduction*

In an unpublished thesis [**9**], G. L. Watson proved that there are only finitely many equivalence classes of positive definite primitive integral ternary quadratic forms that are regular, in the sense that they represent all integers represented by their genus. The methods of that paper produced specific upper bounds for the prime power divisors of the discriminants of such regular ternary forms and described a method that could in principle be used to determine representatives from all equivalence classes of these forms. Watson subsequently published the proof of a more general, but not computationally effective, result establishing the asymptotic growth (with discriminant) of the exceptional set consisting of integers represented by the genus of a positive definite primitive integral ternary quadratic form but not by the form itself [**10**].

More recently, Kaplansky revived interest in the problem of completing the enumeration of the list of regular positive definite primitive integral ternary quadratic forms. His investigations in this direction culminated in a joint paper with Jagy and Schiemann [**7**], in which the authors use the basic method of Watson's thesis, along with extensive machine computation, to produce a list of 913 ternary forms containing a representative from every equivalence class of regular positive definite primitive integral ternary quadratic forms.

In the present paper, we address the analogous problem of determining the positive definite integral ternary quadratic forms that satisfy the weaker property of spinor regularity, that is, those forms that represent all integers represented by their spinor genus. Since the spinor genus is contained in the genus, every regular form is also spinor regular. On the other hand, the list given in [**1**] contains examples of spinor regular ternary quadratic forms that are not regular. Also in [**1**], it is proven that there are only finitely many distinct equivalence classes of spinor regular positive definite primitive integral ternary quadratic forms.

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However, the proof given there relies on the asymptotic methods of [**10**] and consequently does not produce a bound for the largest discriminant of such a form.

Throughout this paper, the geometric language of quadratic spaces and lattices will be adopted. Unexplained notation and terminology will follow that of O'Meara's book [**8**]. For convenience, the term 'ternary lattice' will be used to always refer to a $\mathbb{Z}$-lattice L on a (not necessarily fixed) positive definite ternary quadratic space (V,Q) over the field $\mathbb{Q}$ of rational numbers. For a ternary lattice L, $\operatorname{spn} L$ and $\operatorname{gen} L$ will denote the spinor genus and genus of L, respectively, and $Q(L)$, $Q(\operatorname{spn} L)$, and $Q(\operatorname{gen} L)$ will denote the corresponding sets of represented values. In this notation, L is regular or spinor regular when $Q(L) = Q(\operatorname{gen} L)$ or $Q(L) = Q(\operatorname{spn} L)$, respectively. Note that the property of spinor regularity for L is unaffected by scaling the form Q on V. Consequently, to describe all spinor regular ternary lattices it suffices to consider those with a fixed norm ideal $\mathfrak{n}L$. We will refer to the ternary lattice L as 'normalized' if $\mathfrak{n}L = 2\mathbb{Z}$. The discriminant dL (in the sense of [**8**, § 82B]) of a normalized ternary lattice is an even positive integer; we will denote the quantity $\frac{1}{2}dL$ by δL.

To see the connection with the terminology for quadratic forms used in more classical literature, let $f = \sum_{1\leqslant i\leqslant j\leqslant \ell} a_{ij}x_ix_j$, with $a_{ij} \in \mathbb{Z}$, be an integral quadratic form. Associate to f the matrix $M_f = (\partial^2 f/\partial x_i \partial x_j) = (a'_{ij})$ (that is, $a'_{ii} = 2a_{ii}$ and $a'_{ij} = a'_{ji} = a_{ij}$ for $i < j$). Let V_f be the rational vector space spanned by vectors $e_1, \dots, e_\ell$ equipped with the symmetric bilinear form B_f for which $B_f(e_i, e_j) = a'_{ij}$, and the corresponding quadratic map $Q_f(v) = B_f(v, v)$. Then $L_f = \mathbb{Z}e_1 + \dots + \mathbb{Z}e_\ell$ is a $\mathbb{Z}$-lattice on V_f for which $\mathfrak{n}L_f \subseteq 2\mathbb{Z}$. The original form f is primitive in the sense that the greatest common divisor $\gcd_{1\leqslant i\leqslant j\leqslant \ell}\{a_{ij}\} = 1$ if and only if $\mathfrak{n}L_f = 2\mathbb{Z}$. Such a primitive form f is classically integral if and only if $\mathfrak{s}L_f = 2\mathbb{Z}$; otherwise, $\mathfrak{s}L_f = \mathbb{Z}$ and f is non-classically integral (sometimes referred to as integer-valued). An integer a is represented by the form f if and only if $2a$ is represented by the lattice L_f. Finally, if f is a ternary form and df is the discriminant of f used in [**9**], then $dL_f = \det M_f = 2df$.

The main results obtained in this paper are stated in the following two theorems.

THEOREM 1.1. *Let L be a spinor regular normalized ternary lattice. Then there exists a sequence of spinor regular normalized ternary lattices $L_1, \dots, L_t = L$ such that δL_1 is squarefree and, for $i = 1, \dots, t-1$, there exist primes p_i and integers $k_i \in \{1, 2, 4\}$ such that $dL_{i+1} = p_i^{k_i} dL_i$.*

THEOREM 1.2. *Let L be a spinor regular normalized ternary lattice. Then the prime divisors of dL lie in the set $\mathcal{S} = \{2, 3, 5, 7, 11, 13, 17, 23\}$ and, for each prime $p \in \mathcal{S}$, there exists an explicitly determined positive integer $b(p)$ such that $\operatorname{ord}_p dL \leqslant b(p)$.*

Taken together, these results form the basis for a procedure for producing a list containing all normalized ternary lattices that are potentially spinor regular. The proof of Theorem 1.1 appears in § 3. Explicit bounds for the prime power divisors, as required for Theorem 1.2, are given in Propositions 5.4 and 5.5.

2. *Reducing prime power divisors of the discriminant*

The goals of this section are to define and analyze mappings that can be used to reduce the prime power divisors occurring in the discriminant of a ternary lattice, while preserving regularity properties of the lattice. These mappings are analogous to the transformations used by Watson [**9**]; see also [**4**, **11**] for related constructions. The definition will be based on the following construction for sublattices, which is formulated both for ternary lattices L and their localizations L_p with respect to a prime p.

DEFINITION 2.1. For any ternary lattice L and any $m \in \mathbb{N}$, let $\Lambda_m(L) = \{x \in L : Q(x+z) \equiv Q(z) \pmod{m}, \text{ for all } z \in L\}$ and for any prime p, let $\Lambda_m(L_p) = \{x \in L_p : Q(x+z) \equiv Q(z) \pmod{m}, \text{ for all } z \in L_p\}$. (Equivalently, the defining conditions can be expressed as $Q(x) + 2B(x,z) \in m\mathbb{Z}$ or $m\mathbb{Z}_p$, for all $z \in L$ or L_p, respectively, where B is the bilinear form related to Q by the equation $Q(x+y) = Q(x) + Q(y) + 2B(x,y)$.)

The proofs of the assertions in the following lemma are straightforward consequences of the definitions.

LEMMA 2.2. *Let m be a positive integer and p a prime. Then the following hold.*

(a) *$\Lambda_m(L)$ is a sublattice of L and $\Lambda_m(L_p)$ is a sublattice of L_p.*
(b) *$\Lambda_m(L_p) = (\Lambda_m(L))_p$.*
(c) *$\Lambda_m(L_p) = L_p$ whenever p does not divide m.*
(d) *$\mathfrak{n}(\Lambda_m(L)) \subseteq m\mathbb{Z}$ and $\mathfrak{n}(\Lambda_m(L_p)) \subseteq m\mathbb{Z}_p$.*
(e) *If $\mathfrak{s}L \subseteq \mathbb{Z}$, then $pL \subseteq \Lambda_{2p}(L)$ and $pL_p \subseteq \Lambda_{2p}(L_p)$.*
(f) *If N splits L_p and $\mathfrak{n}N \subseteq 2p\mathbb{Z}_p$, then $N \subseteq \Lambda_{2p}(L_p)$.*
(g) *If $\mathfrak{s}L_2 \subseteq 2\mathbb{Z}_2$, then $\Lambda_4(L_2) = \{x \in L_2 : Q(x) \in 4\mathbb{Z}_2\}$.*

If $\mathfrak{n}L = 2\mathbb{Z}$, it follows from Lemma 2.2(d) and (e) that $2p^2\mathbb{Z} \subseteq \mathfrak{n}(\Lambda_{2p}(L)) \subseteq 2p\mathbb{Z}$. Since equality must hold in one of these containments, it follows that the lattice $\Lambda_{2p}(L)$ can be scaled by either $1/p$ or $1/p^2$ so that the norm ideal of the resulting lattice is $2\mathbb{Z}$, thus producing a new normalized ternary lattice. We denote by λ_p the mapping that sends each normalized ternary lattice on the space V to the normalized ternary lattice constructed in this way on the scaled space $V^{1/p}$ or V^{1/p^2}. That is,

$$\lambda_p(L) = \begin{cases} \Lambda_{2p}(L)^{1/p} & \text{if } \mathfrak{n}(\Lambda_{2p}(L)) = 2p\mathbb{Z} \\ \Lambda_{2p}(L)^{1/p^2} & \text{if } \mathfrak{n}(\Lambda_{2p}(L)) = 2p^2\mathbb{Z}. \end{cases}$$

For the remainder of this section, L will be a normalized ternary lattice and p a prime. For a prime $q \neq p$, it follows from the definition of λ_p and Lemma 2.2(c) that $(\lambda_p(L))_q$ is simply the original lattice L_q scaled by an element of $\mathfrak{u}_q$. Moreover, $(\lambda_p(L))_q \cong L_q$ when the scaling factor in the definition of $\lambda_p(L)$ is $1/p^2$.

In order to determine the structure of $(\lambda_p(L))_p$, it is first necessary to characterize the sublattice $\Lambda_{2p}(L_p)$ of L_p. For this purpose, it is convenient to fix a splitting of L_p as $L_p = M \perp N$, where M is the leading Jordan component of L_p and $\mathfrak{s}N \subsetneq \mathfrak{s}M$. If p is odd, then $\mathfrak{s}M = \mathfrak{s}L_p = \mathfrak{n}L_p = \mathbb{Z}_p$; so M is unimodular. When $p = 2$, either

$\mathfrak{s}M = 2\mathbb{Z}_2$ or $\mathfrak{s}M = \mathbb{Z}_2$. In the first case, M is 2-modular and thus has an orthogonal basis by [**8**, 93:15]. In the second case, it follows from [**8**, 93:15] that M is binary. Then, by [**8**, 93:11], $M \cong \mathbb{H} := A(0,0)$ if M is isotropic, and $M \cong \mathbb{A} := A(2,2)$ if M is anisotropic, where $A(a,b)$ denotes the matrix

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}.$$

LEMMA 2.3. *If M is unimodular and $\mathfrak{n}N \subset 2p\mathbb{Z}_p$, then $\Lambda_{2p}(L_p) = pM \perp N$.*

Proof. The containment $pM \perp N \subseteq \Lambda_{2p}(L_p)$ follows from Lemma 2.2(e) and (f). To prove the reverse containment, let $x \in \Lambda_{2p}(L_p)$ and write $x = x_0 + x_1$, where $x_0 \in M$ and $x_1 \in N$. If $x_0 \notin pM$, then, by [**8**, 82:17], there exists $y \in M$ such that $B(x_0, y) = 1$. Then $Q(x) + 2B(x,y) = Q(x) + 2B(x_0, y) = Q(x) + 2$. However $Q(x) \in 2p\mathbb{Z}_p$ since $x \in \Lambda_{2p}(L_p)$. Thus $Q(x) + 2B(x,y) \notin 2p\mathbb{Z}_p$, contrary to the assumption that $x \in \Lambda_{2p}(L_p)$. Hence $x \in pM \perp N$ as desired. □

LEMMA 2.4. *If $p = 2$ and M is 2-modular, then $[L_2 : \Lambda_4(L_2)] = 2$.*

Proof. In this case, $\mathfrak{s}L_2 = 2\mathbb{Z}_2$, and it follows from Lemma 2.2(g) that $\Lambda_4(L_2) = \{x \in L_2 : Q(x) \in 4\mathbb{Z}_2\}$. Since $\mathfrak{s}N \subsetneq \mathfrak{s}M = 2\mathbb{Z}_2$, it follows that $\mathfrak{n}N \subseteq 4\mathbb{Z}_2$ and so $N \subseteq \Lambda_4(L_2)$. Since $\mathfrak{s}M = 2\mathbb{Z}_2 = \mathfrak{n}L_2 = \mathfrak{n}M$, M has an orthogonal basis by [**8**, 93:15]. We consider the various possible cases for $\dim M$. Suppose first that $\dim M = 1$, so there exists some $u \in M$ such that $Q(u) \in 2\mathfrak{u}_2$ and $M = \mathbb{Z}_2 u$. Let $x = \alpha u + y \in \Lambda_4(L_2)$, where $\alpha \in \mathbb{Z}_2$ and $y \in N$. Then $Q(x) = \alpha^2 Q(u) + Q(y) \in 4\mathbb{Z}_2$. This implies that $\alpha \in 2\mathbb{Z}_2$, since $Q(u) \in 2\mathfrak{u}_2$ and $Q(y) \in 4\mathbb{Z}_2$. Hence $\Lambda_4(L_2) \subseteq 2M \perp N$. Equality then follows from Lemma 2.2(e) and (f), and so $[L_2 : \Lambda_4(L_2)] = 2$ in this case. Next consider the case $\dim M = 2$. Then $M \cong \langle 2a, 2b \rangle$ in some basis $\{u, v\}$, with $a, b \in \mathfrak{u}_2$. Let M' denote the sublattice of M spanned by $\{2u, u+v\}$. Then $M' \perp N \subseteq \Lambda_4(L_2)$, since $Q(2u)$ and $Q(u+v)$ are in $4\mathbb{Z}_2$. Since $\Lambda_4(L_2) \neq L_2$ and $[L_2 : M' \perp N] = 2$, it follows that $M' \perp N = \Lambda_4(L_2)$ and the result is proved in this case. Finally, consider the case $\dim M = 3$ (that is, L_2 is 2-modular). In this case, there is a basis $\{u, v, w\}$ for L_2 with respect to which $L_2 \cong \langle 2a, 2b, 2c \rangle$, where $a, b, c \in \mathfrak{u}_2$. Consider the sublattice M' of L_2 spanned by $\{2u, u+v, v+w\}$. Again we have $M' \subseteq \Lambda_4(L_2) \neq L_2$ and $[L_2 : M'] = 2$, and the result follows. This completes the proof. □

It can now be shown that the mapping λ_p reduces the power of p dividing the discriminant of L whenever this discriminant is divisible by $2p^2$.

LEMMA 2.5. *If $2p^2 | dL$, then $d(\lambda_p(L)) = (1/p^t)dL$ for some $t \in \{1, 2, 4\}$.*

Proof. Note first that $d(\Lambda_{2p}(L)) = p^2 dL$ if either M is unimodular of rank 1 (by Lemma 2.3) or $\mathfrak{s}L = 2\mathbb{Z}$ and $p = 2$ (by Lemma 2.4 and [8, 82:11]). In these cases, $d(\lambda_p(L)) = (1/p)dL$ or $(1/p^4)dL$, depending upon whether $\mathfrak{n}(\Lambda_{2p}(L)) = 2p\mathbb{Z}$ or $2p^2\mathbb{Z}$, respectively. The only remaining case is when M is unimodular of rank 2. Then $\mathfrak{n}N \subseteq 2p^2\mathbb{Z}_p$, by the assumption that $2p^2 | dL$; so $\mathfrak{n}(\Lambda_{2p}(L)) = 2p^2\mathbb{Z}$, by Lemma 2.3. Thus $d(\lambda_p(L)) = (1/p^2)^3 d(\Lambda_{2p}L) = (1/p^6)(p^4 dL) = (1/p^2)dL$. □

The remainder of this section will be devoted to the concrete description of the action of the mapping λ_p for various cases of the splitting L_p.

LEMMA 2.6. *Let L be a normalized ternary lattice such that $L_p \cong \mathbb{H} \perp \langle p^\gamma c\rangle$, for some integer $\gamma \geqslant 3$ and some $c \in \mathbb{Z}_p$. Then $(\lambda_p(L))_p \cong \Lambda_{2p}(L)^{1/p^2} \cong \mathbb{H} \perp \langle p^{\gamma-2}c\rangle$, and $(\lambda_p(L))_q \cong L_q$ for all $q \neq p$.*

Proof. By Lemma 2.3,

$$\Lambda_{2p}(L_p) \cong \begin{pmatrix} 0 & p^2 \\ p^2 & 0 \end{pmatrix} \perp \langle p^\gamma c\rangle.$$

Here $p^\gamma c \in 2p^2\mathbb{Z}_p$ since $\gamma \geqslant 3$, so $\mathfrak{n}(\Lambda_{2p}(L)) = 2p^2\mathbb{Z}$. Thus $(\lambda_p(L))_p = (\Lambda_{2p}(L_p))^{1/p^2} \cong \mathbb{H} \perp \langle p^{\gamma-2}c\rangle$, as claimed. For the primes $q \neq p$, it suffices to note that $(\lambda_p(L))_q = (\Lambda_{2p}(L_q))^{1/p^2} = (L_q)^{1/p^2} \cong L_q$, since $1/p^2 \in \mathfrak{u}_q^2$. □

Now let p be an odd prime. In this case, there is a local splitting of the form

$$L_p \cong \langle a, p^\beta b, p^\gamma c\rangle, \qquad \text{where } 0 \leqslant \beta \leqslant \gamma \text{ are integers and } a, b, c \in \mathfrak{u}_p. \tag{2.1}$$

Applying Lemma 2.3 and the definition of λ_p leads to the following result.

LEMMA 2.7. *Let L be a normalized ternary lattice, let p be an odd prime such that $p^2|dL$, and let (2.1) be a splitting of L_p. Then*

$$(\lambda_p(L))_p \cong \begin{cases} \langle a, b, p^{\gamma-2}c\rangle & \text{when } \beta = 0 \text{ and } \gamma \geqslant 2 \\ \langle b, pa, p^{\gamma-1}c\rangle & \text{when } \beta = 1 \\ \langle a, p^{\beta-2}b, p^{\gamma-2}c\rangle & \text{when } \beta \geqslant 2. \end{cases}$$

For a positive integer k, let λ_p^k denote the k-fold application of the mapping λ_p. Repeated application of Lemma 2.7 then gives the following.

COROLLARY 2.8. *Assume notation as in Lemma 2.7. If $\beta = 1$, then*

$$\left(\lambda_p^\gamma(L)\right)_p \cong \begin{cases} \langle b, c, pa\rangle & \text{when } \gamma \text{ is odd} \\ \langle a, c, pb\rangle & \text{when } \gamma \text{ is even.} \end{cases}$$

If $2 \leqslant \beta < \gamma$ and β_0 denotes the greatest integer in $\beta/2$, then

$$\left(\lambda_p^{\beta_0}(L)\right)_p \cong \begin{cases} \langle a, b, p^{\gamma-\beta}c\rangle & \text{when } \beta \text{ is even} \\ \langle a, pb, p^{\gamma-\beta+1}c\rangle & \text{when } \beta \text{ is odd.} \end{cases}$$

REMARK 2.9. More cases for the possible local structure of the lattice L_2 need to be distinguished in order to obtain the complete description of the action of the mapping λ_2.

3. *Preserving regularity*

Throughout this section, L will denote a normalized ternary lattice. In order to analyze the relationship between regularity properties of L and $\lambda_p(L)$, it is helpful to consider two cases separately, depending upon whether or not the localization L_p is split by a hyperbolic plane.

LEMMA 3.1. *Assume that* $2p^2|dL$ *and that* L_p *is not split by* $\mathbb{H}$. *If* $x \in L_p$ *and* $Q(x) \in 2p\mathbb{Z}_p$, *then* $x \in \Lambda_{2p}(L_p)$.

Proof. It will be convenient to maintain the notation $L_p = M \perp N$ as introduced in the previous section for the p-adic splitting of L. Write $x = x_0 + x_1$, with $x_0 \in M$ and $x_1 \in N$. Then $Q(x) = Q(x_0) + Q(x_1)$. Since $Q(x) \in 2p\mathbb{Z}_p$ and $Q(x_1) \in \mathfrak{n}N \subseteq \mathfrak{s}N \subseteq 2p\mathbb{Z}_p$, it follows that $Q(x_0) \in 2p\mathbb{Z}_p$.

Consider first the case p odd. Then M is unimodular, and by Lemma 2.3, it suffices to show that $x_0 \in pM$. Otherwise, x_0 is a maximal vector of M and, by [**8**, 82:17], there exists $y \in M$ such that $B(x_0, y) = 1$. Then $K = \mathbb{Z}_p x_0 + \mathbb{Z}_p y$ is a binary unimodular lattice of discriminant $dK = (-1)\mathfrak{u}_p^2$. Thus K is a hyperbolic plane, and K splits L_p by [**8**, 82:15], contrary to assumption.

Now consider the case $p = 2$. Suppose first that $\mathfrak{s}M = 2\mathbb{Z}_2$. For $z \in L_2$, write $z = z_0 + z_1$, with $z_0 \in M$ and $z_1 \in N$. Then $Q(x) + 2B(x, z) = Q(x) + 2B(x_0, z_0) + 2B(x_1, z_1)$. Since $\mathfrak{s}N \subset \mathfrak{s}M = 2\mathbb{Z}_2$, we have $2B(x_0, z_0) + 2B(x_1, z_1) \in 4\mathbb{Z}_2$. Hence $Q(x) + 2B(x, z) \in 4\mathbb{Z}_2$ for all $z \in L_2$. That is, $x \in \Lambda_4(L_2)$. Otherwise, $\mathfrak{s}M = \mathbb{Z}_2$. Since $\mathfrak{n}M = 2\mathbb{Z}_2$, M must be binary by [**8**, 93:15], and so either $M \cong \mathbb{H}$ or $M \cong \mathbb{A}$, by [**8**, 93:11]. Since L_2 is not split by a hyperbolic plane, the only possibility is $M \cong \mathbb{A}$. Thus M primitively represents only elements of $2\mathfrak{u}_2$. However $Q(x_0) \in 4\mathbb{Z}_2$, so it must be that $x_0 \in 2M$, and $x \in \Lambda_4(L_2)$ follows from Lemma 2.3. □

PROPOSITION 3.2. *If* L *is spinor regular,* $2p^2|dL$ *and* L_p *is not split by* $\mathbb{H}$, *then* $\lambda_p(L)$ *is spinor regular.*

Proof. For the sake of argument, consider the case in which $\lambda_p(L) = (\Lambda_{2p}(L))^{1/p}$. Let $a \in Q(G)$, where G is a lattice in the spinor genus of $\lambda_p(L)$. Then $pa \in Q(G^p)$, so pa is represented by the spinor genus of $\Lambda_{2p}(L)$, and hence by the spinor genus of L. By the spinor regularity of L, it then follows that there exists $x \in L$ such that $Q(x) = pa \in 2p\mathbb{Z}_p$. Then $x \in \Lambda_{2p}(L)$, by Lemma 3.1. Hence $a \in Q(\Lambda_{2p}(L)^{1/p}) = Q(\lambda_p(L))$. The case in which $\lambda_p(L) = \Lambda_{2p}(L)^{1/p^2}$ is analogous. □

PROPOSITION 3.3. *If* L *is regular,* $2p^2|dL$ *and* L_p *is split by* $\mathbb{H}$, *then* $\lambda_p(L)$ *is regular.*

Proof. Let $a \in Q(\text{gen}\,\lambda_p(L))$; that is, $a \in Q((\lambda_p(L))_q)$ for all q. By Lemma 2.6, $(\lambda_p(L))_q \cong L_q$ for $q \neq p$ so $a \in Q(L_q)$ for all $q \neq p$. Moreover, $a \in \mathfrak{n}((\lambda_p(L))_p) = 2\mathbb{Z}_p$. However $2\mathbb{Z}_p = Q(L_p)$, since L_p is split by $\mathbb{H}$, so $a \in Q(L_q)$ for all q; thus $a \in Q(\text{gen}\,L)$. It follows that $a \in Q(L)$, by regularity of L. Then $p^2 a \in Q(\Lambda_{2p}(L))$, by Lemma 2.2(e). Therefore, $a \in Q(\Lambda_{2p}(L)^{1/p^2}) = Q(\lambda_p(L))$. Hence $\lambda_p(L)$ is regular. □

REMARK 3.4. Note that the argument in the preceding proof is no longer valid when regularity is replaced with the weaker condition of spinor regularity (the local conditions that ensure $a \in Q(\text{gen}\,L)$ do not suffice to imply that $a \in Q(\text{spn}\,L)$). Consequently, some care must be taken when applying a sequence of λ_p mappings to ensure that the resulting lattice always remains spinor regular.

For the purpose of this paper, we will say that 'L behaves well at p' if either $2p^2$ does not divide dL or L_p is split by $\mathbb{H}$.

LEMMA 3.5. *If L behaves well at p, then $Q(L_p) \supseteq 2\mathfrak{u}_p$ and $\theta(O^+(L_p)) \supseteq \mathfrak{u}_p\dot{\mathbb{Q}}_p^2$, where θ denotes the spinor norm mapping.*

Proof. If L_p is split by $\mathbb{H}$, then the results follow since $Q(\mathbb{H}) = 2\mathbb{Z}_p$ and $\theta(O^+(\mathbb{H})) = \mathfrak{u}_p\dot{\mathbb{Q}}_p^2$, by [**8**, 92:5] and [**5**, Lemma 1]. If p is odd and p^2 does not divide dL, then L_p has a unimodular Jordan component of rank at least 2, and the results follow from [**8**, 92:1b and 92:5]. Finally, if $p = 2$ and dL is not divisible by 8, then L_2 has a binary even unimodular Jordan component; otherwise, we would have $\mathfrak{s}L_2 = 2\mathbb{Z}_2$ and it would follow that $\mathfrak{v}L_2 \subseteq (\mathfrak{s}L_2)^3 \subseteq 8\mathbb{Z}_2$, contrary to assumption. Thus L_2 is split by either $\mathbb{H}$ or $\mathbb{A}$. In the latter case, $Q(\mathbb{A}) = 2\mathfrak{u}_2\mathbb{Z}_2^2$ and $\theta(O^+(\mathbb{A})) = \mathfrak{u}_2\dot{\mathbb{Q}}_2^2$ by [**5**, Lemma 1]. □

COROLLARY 3.6. *If L behaves well at all primes, then* $\operatorname{spn} L = \operatorname{gen} L$.

Proof. The proof follows from [**8**, 102:9]. □

Proof of Theorem 1.1. By Lemma 2.5, it suffices to show that for any spinor regular ternary lattice L for which δL is not squarefree, there exists some prime p for which $2p^2|dL$ and $\lambda_p(L)$ is spinor regular. If there exists a prime p such that $2p^2|dL$ and L_p is not split by $\mathbb{H}$, then $\lambda_p(L)$ is spinor regular by Proposition 3.2. If there is no such prime, then L behaves well at q for all primes q. Then $\operatorname{spn} L = \operatorname{gen} L$ by Corollary 3.6. Thus L is regular, and, by Proposition 3.3, $\lambda_p(L)$ is regular (hence spinor regular) for any prime p such that $2p^2|dL$. □

REMARK 3.7. It may be of interest to observe that the statements and proofs of Proposition 3.2 and Theorem 1.1 carry over verbatim with the condition of spinor regularity replaced by that of regularity. It seems that this result has not appeared explicitly in the literature, although it is proven in Watson's thesis [**9**] and is used for the search conducted to produce the list appearing in [**7**].

4. *A method of descent*

The remainder of the paper will be devoted to proving Theorem 1.2. The argument will first be reduced to the case of normalized ternary lattices for which all localizations, except possibly for one, have a particularly simple structure.

For a normalized ternary lattice L and a prime p, we will say that 'L behaves well away from p' if L behaves well at q for all primes $q \neq p$.

LEMMA 4.1. *Let L be a spinor regular normalized ternary lattice. For any prime p there exists a spinor regular normalized ternary lattice L' such that L' behaves well away from p and $\operatorname{ord}_p dL = \operatorname{ord}_p dL'$.*

Proof. Let $q \neq p$ be a prime such that $2q^2|dL$. If $\mathbb{H}$ splits L_q, then L behaves well at q. Otherwise, $\lambda_q(L)$ is spinor regular by Proposition 3.2. Moreover, $\operatorname{ord}_q d(\lambda_q(L)) = \operatorname{ord}_q dL - 2$ and $(\lambda_q(L))_r \cong L_r$ for all primes $r \neq q$ by Lemma 2.6.

Thus applying λ_q to L a finite number of times leads to a spinor regular normalized ternary lattice that behaves well at q. Repeating this procedure for all primes $q \neq p$ such that $2q^2|dL$ leads to a lattice L' with the desired properties. □

Let L be a normalized ternary lattice that behaves well away from some prime p. Repeatedly apply λ_p to L until a lattice is obtained for which the p-adic localization is split by $\mathbb{H}$ or for which the discriminant is not divisible by $2p^2$, whichever occurs first. Denote this lattice by $\tilde{L}$. Next, for each prime q such that $2q^2|d\tilde{L}$ (including the possibility that $q = p$), repeatedly apply λ_q to $\tilde{L}$ until a lattice is obtained for which the discriminant is no longer divisible by $2q^2$. When this has been completed for all primes q, denote the resulting lattice by $\hat{L}$. Note that this construction ensures that $\delta\hat{L}$ is squarefree.

LEMMA 4.2. *Let L be a spinor regular normalized ternary lattice that behaves well away from the prime p and let $\tilde{L}$ and $\hat{L}$ be as in the construction above. Then, the following hold.*

(a) *$\tilde{L}$ and $\hat{L}$ are regular.*

(b) *$\mathrm{ord}_q d\hat{L} \equiv \mathrm{ord}_q dL \pmod 2$ holds for all $q \neq p$.*

(c) *$Q(L_q) = 2\mathbb{Z}_q$ for any prime $q \neq p$ which does not divide $\delta\hat{L}$.*

(d) *$2q \in Q(\mathrm{gen}\, L)$ for any prime $q \neq p$ such that q does not divide $\delta\hat{L}$ and $2q \in Q(L_p)$.*

Proof. (a) For each application of λ_p in the construction of $\tilde{L}$, spinor regularity is preserved, by Proposition 3.2. Thus, $\tilde{L}$ is spinor regular. Moreover, the construction ensures that $\tilde{L}$ behaves well at all primes. Hence $\mathrm{spn}\,\tilde{L} = \mathrm{gen}\,\tilde{L}$ by Corollary 3.6, and so $\tilde{L}$ is regular. For each prime q such that $2q^2|d\tilde{L}$, L_q is split by $\mathbb{H}$, so by Proposition 3.3, it follows that $\hat{L}$ is regular.

(b) The mapping λ_q is applied in the construction of $\hat{L}$ only when L_q is split by $\mathbb{H}$, in which case the discriminant is reduced by a factor of q^2, by Lemma 2.6.

(c) Since $\mathrm{ord}_q\delta\hat{L} = 0$, $\mathrm{ord}_q\delta L$ must be even. If $\mathrm{ord}_q\delta L \geqslant 2$, then L_q is split by $\mathbb{H}$, since L behaves well at q, and the result follows. Thus only $\mathrm{ord}_q\delta L = 0$ remains. If q is odd, then L_q is unimodular and the result follows from [**8**, 92:1b]. If $q = 2$, then L_2 has a splitting of the form $\mathbb{H} \perp \langle 2c\rangle$, for some $c \in \mathfrak{u}_2$, and the argument is complete.

(d) The local representability of $2q$ follows from (c) for the prime q, and from Lemma 3.5 for the primes distinct from p and q. □

PROPOSITION 4.3. *Let p be a prime that divides the discriminant of some spinor regular normalized ternary lattice. Then p divides the discriminant of some regular normalized ternary lattice.*

Proof. For such a prime p, there exists a spinor regular normalized ternary lattice L for which $p|dL$ and L behaves well away from p, by Lemma 4.1. It suffices to consider the case when p is odd. Thus L_p has a splitting of the type (2.1). If either $\beta = 0$ or $\beta = \gamma$ in such a splitting, then $\theta(O^+(L_p)) \supseteq \mathfrak{u}_p\dot{\mathbb{Q}}_p^2$ by [**8**, 92:5]. Thus, $\mathrm{spn}\, L = \mathrm{gen}\, L$ by [**8**, 102:9], and L is itself regular. If $\beta = 1$, then $\tilde{L} = \lambda_p^{\gamma}(L)$; so $p|d\tilde{L}$, by Corollary 2.8, and the result follows since $\tilde{L}$ is regular by Lemma 4.2(a). It remains to consider the case $2 \leqslant \beta < \gamma$. In that case, $M = \lambda_p^{\beta_0}(L)$ is spinor regular, where β_0 denotes the greatest integer in $\beta/2$, by the proof of Lemma 4.2(a). If β is even, it can be seen from Corollary 2.8 that M is regular (here $\operatorname{spn} M = \operatorname{gen} M$ because M_p is split by a binary unimodular sublattice) and $p|dM$ (since $\beta \neq \gamma$). If β is odd, then $\tilde{M}$ is regular and $p|d\tilde{M}$. □

Via the correspondence described in § 1, the prime divisors of the discriminants of regular normalized ternary lattices correspond to the prime divisors of the discriminants of regular primitive integral ternary quadratic forms. These prime divisors are determined in [**9**, Theorems 2, 3 and 4] to be 2, 3, 5, 7, 11, 13, 17 and 23. (This result can also be verified by an examination of the factorizations of the discriminants of the forms listed in [**7**].)

Corollary 4.4. *Let p be a prime that divides the discriminant of some spinor regular normalized ternary lattice. Then $p \in \mathcal{S} = \{2, 3, 5, 7, 11, 13, 17, 23\}$.*

The regular normalized ternary lattices L for which δL is squarefree correspond to the regular primitive integral ternary quadratic forms of squarefree discriminant. The complete list of such forms is given in [**12**]. For convenience of reference, the list of the corresponding lattices, along with their discriminants and the structures of their localizations at all prime divisors of the discriminant, is given in Appendix A. Note that for any spinor regular normalized ternary lattice L the corresponding lattice $\hat{L}$ is among the lattices in this list.

5. *Bounds for prime power divisors*

For the purpose of bounding the powers of primes dividing the discriminants of spinor regular normalized ternary lattices, it suffices to establish bounds for the successive minima $\mu_i(L)$ for such lattices, in light of the fundamental inequality $dL \leqslant \mu_1(L)\mu_2(L)\mu_3(L)$ (see [**2**]). As a first step in the direction of establishing such bounds, Lemma 4.2 can be used to identify integers represented by the genus of L. In order to conclude that such an integer n is represented by L itself when L is spinor regular, it must be established that n is represented by the spinor genus of L. This is ensured when it can be shown that n is not a spinor exception for $\operatorname{gen} L$, that is, that n is represented by every spinor genus in $\operatorname{gen} L$. The following assertion will be used for that purpose.

Lemma 5.1. *Suppose that the normalized ternary lattice L behaves well away from the prime p and that n is a spinor exception for* $\operatorname{gen} L$. *Then $\mathbb{Q}(\sqrt{-ndL})$ is a quadratic extension of $\mathbb{Q}$ and*

$$\mathbb{Q}(\sqrt{-ndL}) \subseteq \begin{cases} \mathbb{Q}(\sqrt{p^*}) & \text{where } p^* = (-1)^{(p-1)/2}p, \text{ if } p \text{ is odd} \\ \mathbb{Q}(\sqrt{-1}, \sqrt{-2}) & \text{if } p = 2. \end{cases}$$

Proof. Let Σ_L denote the spinor class field of L, as defined in [**3**]. A necessary condition for n to be a spinor exception for $\operatorname{gen} L$ is that $\mathbb{Q}(\sqrt{-ndL})$ is a quadratic subextension of Σ_L (see, for example, [**6**]). Since L behaves well away from p, $\theta(O^+(L_q)) \supseteq \mathfrak{u}_p\dot{\mathbb{Q}}_q^2$ holds for all $q \neq p$. Consequently, Σ_L is a multiquadratic extension of $\mathbb{Q}$ which is unramified at all primes $q \neq p$. □

Corollary 5.2. *If the normalized ternary lattice L behaves well away from p for some prime $p \equiv 1 \pmod 4$, then* gen L *has no spinor exceptions.*

Proof. Since $p \equiv 1 \pmod 4$, $\mathbb{Q}(\sqrt{p^*}) = \mathbb{Q}(\sqrt{p})$ is a real quadratic field. However, for a positive definite lattice L, $\mathbb{Q}(\sqrt{-ndL})$ cannot be a real quadratic field, since n and dL are positive. □

Every normalized ternary lattice L has a reduced basis, that is, a basis $\{e_1, e_2, e_3\}$ such that $Q(e_i) = \mu_i(L)$, the ith successive minimum of L, for each $i = 1, 2, 3$. For such a basis, $2|B(e_i, e_j)| \leqslant Q(e_i)$ holds whenever $i < j$. The sublattice $\mathbb{Z}e_1 + \mathbb{Z}e_2$ will be referred to as the leading binary section of L and its discriminant will be denoted by d_0L. Note that d_0L has the following properties: (a) $d_0L < \mu_1(L)\mu_2(L)$, (b) $d_0L \equiv 0$ or $3 \pmod 4$, and (c) $\beta \leqslant \text{ord}_p d_0L$ when p is odd and L_p has the splitting (2.1).

We are now in a position to establish upper bounds for the powers of the primes occurring in the discriminant of a spinor regular normalized ternary lattice. The results obtained are based upon corresponding bounds which are known for the regular case. For each odd prime $p \in \mathcal{S}$, a bound for the power of p occurring in the discriminant of a regular primitive integral ternary quadratic form appears in [**9**, Theorem 3, 4 or 5]. The results for corresponding normalized ternary lattices are summarized in the following proposition.

Proposition 5.3. *Suppose that $p|dL$ for some regular normalized ternary lattice L and some odd prime p. Let L_p have the splitting* (2.1).

(a) *If $\beta = 0$, then*

$$\gamma \leqslant \begin{cases} 1 & \text{if } p = 7, 11, 13, 17 \text{ or } 23 \\ 2 & \text{if } p = 5 \\ 4 & \text{if } p = 3. \end{cases} \tag{5.1}$$

(b) *If $\beta = 1$, then*

$$\gamma \leqslant \begin{cases} 1 & \text{if } p = 11, 13, 17 \text{ or } 23 \\ 2 & \text{if } p = 5 \text{ or } 7 \\ 4 & \text{if } p = 3. \end{cases} \tag{5.2}$$

(c) *If $\beta \geqslant 2$, then $p = 3$, $\beta \leqslant 3$ and $\beta + \gamma \leqslant 6$.*

The corresponding bounds for spinor regular normalized ternary lattices are given in the following proposition.

Proposition 5.4. *Suppose that $p|dL$ for some spinor regular normalized ternary lattice L and some odd prime p. Let L_p have the splitting* (2.1).

(a) *If $\beta = 0$, then* (5.1) *holds.*

(b) *If $\beta = 1$, then* (5.2) *holds, except possibly when $p = 11$, in which case $\gamma \leqslant 2$.*

(c) *If $\beta \geqslant 2$, then $p = 3$ or 7, and*

$$\begin{cases} \beta = 2 \text{ and } \gamma = 3 & \text{for } p = 7, \\ \beta \leqslant 5 \text{ and } \gamma \leqslant 8 & \text{for } p = 3. \end{cases}$$

Proof. Throughout the proof, it may be assumed that L behaves well away from p, by Lemma 4.1. Whenever it can be shown that L is regular, the desired result follows from Proposition 5.3. In particular, if $\beta = 0$ or $\beta = \gamma$, then $\operatorname{spn} L = \operatorname{gen} L$, and L is regular. This establishes (a). Moreover, if $p \equiv 1 \pmod 4$, $\operatorname{gen} L$ has no spinor exceptions by Corollary 5.2 and again L is regular, so for the remainder of the proof, it will be assumed that

$$p \equiv 3 \pmod 4 \quad \text{and} \quad 1 \leqslant \beta < \gamma.$$

Since it may also be assumed that $p \in \mathcal{S}$, this restricts attention to the primes 3, 7, 11 and 23.

(b) By Corollary 2.8, $\hat{L}_p \cong \langle b, c, pa \rangle$ if γ is odd, and $\hat{L}_p \cong \langle a, c, pb \rangle$ if γ is even. Thus $p | d\hat{L}$ in any case, and if $\hat{L}_p$ is split by a binary sublattice $\langle 1, \Delta_p \rangle$ (where Δ_p denotes a nonsquare unit modulo p), then $\theta(O^+(L_p)) \supseteq \mathfrak{u}_p \dot{\mathbb{Q}}_p^2$ and L is regular. The remaining details of the argument will be presented only for the case $p = 11$, since the arguments for the other primes are analogous and the results for those primes agree with the regular case. The possibilities for $\hat{L}$ with $11 | d\hat{L}$ are #14, #18 and #21 in Appendix A. In the case of #14 or #18, $\hat{L}$ has a unimodular Jordan component $\langle 1, \Delta_{11} \rangle$. Thus it remains to consider the case that $\hat{L}$ is #21. Then $\delta \hat{L} = 11$ and $\hat{L}_{11} \cong \langle 1, 1, 11 \rangle$, so $L_{11} \cong \langle 1, 11, 11^\gamma \rangle$, and $Q(L_q) = 2\mathbb{Z}_q$ for all $q \neq 11$. In particular, $\operatorname{gen} L$ represents 4 and 12, but none of 2, 6, 8 or 10 (since these values are not in $\mathfrak{u}_{11}^2$). Assume for the moment that γ is odd. Then $\operatorname{ord}_{11} dL$ is even and so $\operatorname{ord}_{11} n$ must be odd for any spinor exception n of $\operatorname{gen} L$ by Lemma 5.1. In particular, neither 4 nor 12 is a spinor exception. Thus 4 and 12 are represented by L, so $\mu_1(L) = 4$ and $\mu_2(L) = 4$ or 12. Let $\mathcal{B}$ denote the leading binary section of L. If $\mu_2(L) = 4$, then

$$\mathcal{B} \cong \begin{pmatrix} 4 & t \\ t & 4 \end{pmatrix} \qquad \text{for } t = 0,\ 1 \text{ or } 2.$$

However, all of these possibilities for $\mathcal{B}$ are unimodular over $\mathbb{Z}_{11}$, contrary to the assumption that $\beta = 1$, which implies that $11 | d\mathcal{B}$, so $\mu_2(L) = 12$, and $\mathcal{B}$ must be of the form

$$\begin{pmatrix} 4 & t \\ t & 12 \end{pmatrix} \qquad \text{for } t = 0,\ 1 \text{ or } 2.$$

The first two possibilities are ruled out as they are again unimodular over $\mathbb{Z}_{11}$, so

$$\mathcal{B} \cong \begin{pmatrix} 4 & 2 \\ 2 & 12 \end{pmatrix}.$$

Now $14 \in Q(L)$, but $14 \notin Q(\mathcal{B})$. Thus, $\mu_3(L) \leqslant 14$, and $dL \leqslant 14 d\mathcal{B} = 616 < 11^3$. Recalling that γ has been assumed to be odd, it then follows that $\gamma = 1$. On the other hand, if γ is even, then $\lambda_{11}(L)$ is spinor regular and $(\lambda_{11}(L))_{11} \cong \langle a, 11b, 11^{\gamma-1}c \rangle$. Hence $\gamma \leqslant 2$, as claimed.

(c) Suppose first that $p \geqslant 7$. If β is even, then $\langle a, b, p^{\gamma-\beta}c \rangle$ is spinor regular, by Corollary 2.8, and hence is regular, so $\gamma - \beta \leqslant 1$ by (5.1). However, we are assuming that $\gamma \neq \beta$, so $\gamma = \beta + 1$ and $\hat{L}_p \cong \langle a, b, pc \rangle$. If β is odd, then $\langle a, pb, p^{\gamma-\beta+1}c \rangle$ is spinor regular, again by Corollary 2.8. From the previous case, this forces $\gamma - \beta + 1 \leqslant 2$ and $p \neq 23$, so we again obtain $\gamma = \beta + 1$ and $\hat{L}_p \cong \langle a, c, pb \rangle$ for $p = 7, 11$.

$p = 23$: As noted above, β must be even and $23 | d\hat{L}$, so #19 is the only possibility in Appendix A for $\hat{L}$. Thus, $d\hat{L} = 92 = 4{\cdot}23$, so $\operatorname{ord}_2 dL$ is even, by Lemma 4.2(b). By Lemma 5.1, any possible spinor exception for $\operatorname{gen} L$ must be a multiple of 4. By

TABLE 1.

$\hat{L}$	#5	#8	#11	#15	#16	#24
$d\hat{L}$	4·7	2·7	4·3·7	2·3·7	4·5·7	2·3·7
Form	k^2	$2k^2$	$3k^2$	$6k^2$	$5k^2$	$6k^2$

Lemmas 3.4 and 4.2(d), gen L represents either both of 2 and 2·3 (if $a \in \mathfrak{u}_{23}^2$) or both of 2·5 and 2·7 (if $a \notin \mathfrak{u}_{23}^2$). As none of these integers are spinor exceptions for gen L, it then follows that one of these pairs is represented by spn L, and hence by L itself, by the assumption of spinor regularity. Thus, $d_0L \leqslant \mu_1(L)\mu_2(L) \leqslant 10{\cdot}14 = 140 < 23^2$, contrary to the assumption that $\beta > 1$. Hence, there are no such spinor regular ternary lattices.

$p = 11$: Since $11|d\hat{L}$, $\hat{L}$ must be #14, #18 or #21 in Appendix A. In these cases, the possible spinor exceptions for gen L which are not multiples of 11 are of the forms $6k^2$, k^2, or $2k^2$, respectively, by Lemma 4.2(b) and Lemma 5.1. If $\hat{L}$ is #21, then $\delta\hat{L} = 11$ and $\hat{L}_{11} \cong \langle 1, 1\rangle \perp \langle 11\rangle$. Thus $a \in \mathfrak{u}_{11}^2$ and gen L represents $2q$ for all primes $q \in \mathfrak{u}_{11}^2$. In particular, 2·3 and 2·5 are in $Q(\text{gen } L)$, but neither 6 nor 10 is a spinor exception; so $6, 10 \in Q(\text{spn } L) = Q(L)$. Thus $\mu_1(L) \leqslant 6$ and $\mu_2(L) \leqslant 10$. Therefore, $d_0L \leqslant 60$, which contradicts the fact that $11^2|d_0L$. Now consider the cases when $\hat{L}$ is #14 or #18. In each case, $\hat{L}_{11}$ is split by $\langle 1, \Delta_{11}\rangle$. If β is even, then $\hat{L}_{11} \cong \langle a, b, pc\rangle$ as seen above. It then follows that $\theta(O^+(L_{11})) \supseteq \mathfrak{u}_{11}\dot{\mathbb{Q}}_{11}^2$; so L would be regular, but no such regular ternary lattices exist, by Proposition 5.3. Consider further the case that β is odd; thus $\beta \geqslant 3$. By Lemma 3.5 and Lemma 4.2(d), gen L represents either both of 2 and 2·5 (if $a \notin \mathfrak{u}_{11}^2$), or both of 2·7 and 2·13 (if $a \in \mathfrak{u}_{11}^2$). None of these integers are spinor exceptions for gen L, so one of these pairs is represented by L. Thus $d_0L \leqslant 364$, which contradicts the fact that $11^3|d_0L$. Finally, we conclude that there are no such spinor regular ternary lattices.

$p = 7$: Since $7|d\hat{L}$, the possibilities for $\hat{L}$ are shown in Table 1, which also lists $d\hat{L}$ and the form of any possible spinor exceptions for gen L which are not multiples of 7, in each case.

Suppose first that $a \in \mathfrak{u}_7^2$. Then $2, 22 \in Q(L_7)$. It can be seen from Table 1 that 22 is never a spinor exception for gen L, and 2 can be a spinor exception for gen L only when $\hat{L}$ is #8. In the latter case, $\hat{L}_2$ is split by $\mathbb{H}$, but then L_2 must also be split by $\mathbb{H}$, and so L_2 represents 4. Then $4 \in Q(\text{gen } L)$ and it follows that $4 \in Q(L)$. Thus in all cases $\mu_1(L) \leqslant 4$ and $\mu_2(L) \leqslant 22$, so $d_0L \leqslant \mu_1(L)\mu_2(L) \leqslant 88$. However $\beta \geqslant 2$ implies that $7^2|d_0L$. Moreover, $d_0L \equiv 3 \pmod 4$, so $d_0L \geqslant 3{\cdot}49 = 147 > 88$, a contradiction.

Now consider the case $a \in \Delta_7\mathfrak{u}_7^2$. Then $2{\cdot}3, 2{\cdot}5 \in Q(L_7)$. If $\hat{L}$ is #5, #8 or #16, then $\delta\hat{L}$ is not divisible by 3, and so $6 \in Q(L_3)$. Thus, in these cases, $6 \in Q(\text{gen } L)$. However, 6 is not a spinor exception for gen L for any of the possibilities for $\hat{L}$, so $6 \in Q(L)$. On the other hand, if $\hat{L}$ is one of #11, #15 or #24, then $\delta\hat{L}$ is not divisible by 5 and so $10 \in Q(L_5)$; thus, $10 \in Q(\text{gen } L)$, but 10 is not a spinor exception for gen L in any of these cases, so $10 \in Q(L)$. In all cases, we conclude that $\mu_1(L) \leqslant 10$. Also, $2{\cdot}13 \in Q(\text{gen } L)$. As 26 is not a possible spinor exception for gen L for any of the possibilities for $\hat{L}$, it follows that $26 \in Q(L)$ and $\mu_2(L) \leqslant 26$. Hence, $\mu_1(L)\mu_2(L) \leqslant 260 < 7^3$ and $\beta \leqslant 2$, as asserted. Consequently, $\beta = 2$ and $\gamma = \beta + 1 = 3$.

$p = 3$: Consider first the case that $a \in \Delta_3 \mathfrak{u}_3^2$. Then $2 \in Q(\operatorname{gen} L)$. If 2 is a spinor exception for $\operatorname{gen} L$, then $\mathbb{Q}(\sqrt{-2dL}) = \mathbb{Q}(\sqrt{-3})$ and, in particular, $\operatorname{ord}_7 dL$ is even. Since L behaves well at 7, it follows that L_7 is split by $\mathbb{H}$. Hence, $2{\cdot}7 \in Q(L_7)$ and $14 \in Q(\operatorname{gen} L)$, but 14 is not a spinor exception for $\operatorname{gen} L$ since $\operatorname{ord}_7(-14dL)$ is odd. Thus at least one of 2 and 14 is represented by L. Thus $\mu_1(L) \leqslant 14$. Next, note that $2{\cdot}19 \in Q(\operatorname{gen} L)$, since $19 \notin S$. As $\operatorname{ord}_{19}(-38dL)$ is odd, it follows that 38 is not a spinor exception for $\operatorname{gen} L$ and so $38 \in Q(L)$, so $\mu_2(L) \leqslant 38$. Then $d_0 L \leqslant 532 < 3^6$. Consequently, $\beta \leqslant 5$.

Now consider the case $a \in \mathfrak{u}_3^2$. An examination of the discriminants of the lattices appearing in Appendix A shows that it cannot be the case that the primes 5 and 11 simultaneously divide $d\hat{L}$; so at least one of these primes, denote it by q, fails to divide $d\hat{L}$ and it follows that $2q \in Q(\operatorname{gen} L)$. Moreover, $\operatorname{ord}_q(-2qdL)$ is odd by Lemma 4.2(b), and so $2q \in Q(L)$. Consequently, $\mu_1(L) \leqslant 22$. Repeating this argument with the pair of primes 17 and 23 yields $\mu_2(L) \leqslant 46$. Hence, $d_0 L \leqslant 1012 < 3^7$, and so $\beta \leqslant 6$. Suppose that $\beta = 6$. Then $3^6 | d_0 L$. Since $d_0 L \equiv 0$ or $3 \pmod 4$, it follows that $d_0 L \geqslant 3^7$, a contradiction; so again in this case, $\beta \leqslant 5$, as claimed. The inequality $\gamma \leqslant 8$ then follows either from (5.1) (if β is even) or (5.2) (if β is odd). □

PROPOSITION 5.5. *Let L be a spinor regular normalized ternary lattice.*

(a) *If L_2 is split by $\mathbb{H}$ or $\mathbb{A}$, then $\operatorname{ord}_2 dL \leqslant 7$.*

(b) *Let $L_2 \cong \langle 2a, 2^{\beta+1}b, 2^{\gamma+1}c\rangle$ with $\beta, \gamma \in \mathbb{Z}$ such that $1 \leqslant \beta \leqslant \gamma$ and $a, b, c \in \mathfrak{u}_2$. If $\beta \leqslant 1$, then $\gamma \leqslant 8$. In general, $\beta \leqslant 9$ and $\gamma \leqslant 16$. Consequently, $\operatorname{ord}_2 dL \leqslant 28$.*

Outline proof. Without loss of generality, it may be assumed that L behaves well away from 2. If L_2 is split by $\mathbb{H}$ or $\mathbb{A}$, then $\theta(O^+(L_2)) \supseteq \mathfrak{u}_2 \dot{\mathbb{Q}}_2^2$, as noted previously. Consequently, L is regular and the result stated in (a) follows from [**9**, Theorem 3]. It remains to consider those L for which L_2 has an orthogonal splitting as given in the statement of (b). In future arguments, it suffices to consider a, b, and c to be integers modulo 8 in such a splitting. The proof proceeds by considering three cases for the size of β: (i) $\beta = 0$, (ii) $\beta = 1$, and (iii) β arbitrary. Moreover, it may be assumed throughout that $\gamma \geqslant 5$, since otherwise there is nothing to prove.

Case (i): In this case, there exists a basis $\{u, v, w\}$ in which $L_2 \cong \langle 2a, 2b, 2^{\gamma+1}c\rangle$, for some $a, b, c \in \mathfrak{u}_2$. Then $\Lambda_4(L_2)$ is spanned by $\{2u, u+v, w\}$, as shown in the proof of Lemma 2.4. Thus $\Lambda_4(L_2) \cong N \perp \langle 2^{\gamma+1}c\rangle$, where

$$N \cong \begin{pmatrix} 8a & 4a \\ 4a & 2(a+b) \end{pmatrix}.$$

If $ab \equiv 3 \pmod 4$, then $a+b \in 4\mathbb{Z}_2$, and $\mathfrak{n}(\Lambda_4(L)) = 8\mathbb{Z}$ (since $\gamma \geqslant 3$); so $\lambda_2(L_2) = \Lambda_4(L_2)^{1/4}$. Hence $\lambda_2(L_2) \cong N^{1/4} \perp \langle 2^{\gamma-1}c\rangle$. Here $N^{1/4}$ is a unimodular lattice with $\mathfrak{n}(N^{1/4}) = 2\mathbb{Z}_2$; thus, $N^{1/4} \cong \mathbb{H}$ or $N^{1/4} \cong \mathbb{A}$ by [8, 93:11]. It then follows from part (a) that $\gamma - 1 \leqslant 7$.

Only the subcase $ab \equiv 1 \pmod 4$ needs to be considered further. Note that this condition is equivalent to the condition $a + b \in 2\mathfrak{u}_2$. In this case, $\mathfrak{n}(\Lambda_4(L)) = 4\mathbb{Z}$ and so $\lambda_2(L_2) = \Lambda_4(L_2)^{1/2}$. Thus $\lambda_2(L_2) \cong N^{1/2} \perp \langle 2^\gamma c\rangle \cong \langle 2a', 2b', 2^\gamma c\rangle$, for some $a', b' \in \mathfrak{u}_2$ again satisfying $a'b' \equiv 1 \pmod 4$. Repeatedly applying λ_2 a total of γ times results in a lattice K for which $K_2 \cong \langle 2d, 2e, 2f\rangle$, with $d, e, f \in \mathfrak{u}_2$. Applying λ_2 to K then yields the lattice $\hat{L}$. As can be seen from the proof of Lemma 2.4,

$\hat{L}_2 \cong \mathbb{P} \perp \langle 4\epsilon \rangle$ for some $\epsilon \in \mathfrak{u}_2$, where $\mathbb{P} \cong \mathbb{H}$ or $\mathbb{P} \cong \mathbb{A}$. By examining the 2-adic splittings of the lattices in Appendix A, we see that the resulting $\hat{L}$ must be one of the lattices

$$\#1, \#4, \#5, \#6, \#7, \#9, \#11, \#12, \#16, \#17, \#18, \#19, \#20. \tag{5.3}$$

We temporarily assume that γ is odd. Then L_p is split by $\mathbb{H}$ for the odd primes p for which $p^2|dL$, and $L_p \cong \hat{L}_p$ for all other odd primes p.

Now note that $\theta(O^+(L_2)) \supseteq \theta(O^+(\langle a, b \rangle)) = \mathbb{N}(\mathbb{Q}_2(\sqrt{-ab}))$, by [**5**, Proposition B], where $\mathbb{N}$ denotes the norm mapping from $\mathbb{Q}_2(\sqrt{-ab})$ to $\mathbb{Q}_2$. The remainder of the argument is divided into the subcases $ab \equiv 5 \pmod 8$ and $ab \equiv 1 \pmod 8$. In the subcase that $ab \equiv 5 \pmod 8$, it follows from the above spinor norm calculation that the spinor class field Σ_L must equal $\mathbb{Q}$. Hence L is regular. There are two possibilities to consider: $\langle a, b \rangle \cong \langle 1, 5 \rangle$ and $\langle a, b \rangle \cong \langle -1, 3 \rangle$ over $\mathbb{Z}_2$. Suppose first that $\langle a, b \rangle \cong \langle 1, 5 \rangle$. Then $2 \in Q(\operatorname{gen} L)$; hence $2 \in Q(L)$ and $\mu_1(L) = 2$. By examining the 5-adic splittings of the lattices in (5.3), we see that $10 \in Q(\hat{L}_5)$, hence in $Q(L_5)$. Thus $10 \in Q(\operatorname{gen} L)$ and it follows that $10 \in Q(L)$ and $\mu_2(L) \leqslant 10$. The leading binary section $\mathcal{B}$ of L must then be of the form $\langle 2, 2t \rangle$ for $t =1$, 2, 3, 4 or 5. However $t = 2$ and $t = 3$ are ruled out since $\langle 2, 4 \rangle$ and $\langle 2, 6 \rangle$ do not represent 10; $t = 4$ is ruled out since L_2 contains a 2-modular sublattice of rank 2, by assumption. Hence $\mathcal{B} \cong \langle 2, 2 \rangle$ or $\langle 2, 10 \rangle$. If $13|dL$, then $13|d\hat{L}$, by Lemma 4.2(b), and $\hat{L}$ would have to be #13 or #20. For both of these possibilities, $\mathbb{H}$ splits $\hat{L}_{13}$. Thus, in all cases, $26 \in Q(\operatorname{gen} L)$ and hence $26 \in Q(L)$, but 26 is not represented by $\mathcal{B}$. Hence $\mu_3(L) \leqslant 26$ and so $dL \leqslant 520 < 2^{10}$. We conclude that $\gamma \leqslant 5$ whenever $\langle a, b \rangle \cong \langle 1, 5 \rangle$. A similar analysis when $\langle a, b \rangle \cong \langle -1, 3 \rangle$ also leads to the conclusion that $\gamma \leqslant 5$. This completes the subcase $ab \equiv 5 \pmod 8$.

Finally, consider the subcase $ab \equiv 1 \pmod 8$. Under this condition, $\Sigma_L = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-1})$. Since γ is assumed to be odd, $\operatorname{ord}_2 dL$ is even. In particular, if the integer n is a spinor exception of $\operatorname{gen} L$, then $\operatorname{ord}_2 n$ must be even. The possibilities for $\langle a, b \rangle$ up to isometry over $\mathbb{Z}_2$ are now $\langle 1, 1 \rangle$ and $\langle 3, 3 \rangle$. Suppose that $\langle a, b \rangle \cong \langle 1, 1 \rangle$. Again $\mu_1(L) = 2$. If $\hat{L}$ is not #1, $\operatorname{ord}_p dL$ is odd for at least one odd prime. Then 4 is not a spinor exception, and so $\mu_2(L) \leqslant 4$. The only possible leading binary section for L is $\mathcal{B} \cong \langle 2, 2 \rangle$ ($\langle 2, 4 \rangle$ is not possible since L_2 contains a 2-modular sublattice of rank 2). At least one of 66 and 114 is represented by $\operatorname{gen} L$. However, neither of these values is represented by $\mathcal{B}$ and neither can be a spinor exception for $\operatorname{gen} L$ (for any prime p not dividing $d\hat{L}$, $\operatorname{ord}_p n$ must be even for any spinor exception n; for lattices in (5.3), no discriminant is divisible by 19, or by both 3 and 11). Hence $\mu_3(L) \leqslant 114$ and $dL \leqslant 1824 < 2^{11}$ when $\hat{L}$ is not #1. When $\hat{L}$ is #1, it can be shown that $10 \in Q(L)$ and so $\mu_2(L) \leqslant 10$. The possible leading binary sections can then be shown to be $\langle 2, 2 \rangle$ or $\langle 2, 8 \rangle$. In either case, $\mu_3(L) \leqslant 66$, and it follows that $\gamma \leqslant 5$. Thus it has been established that $\gamma \leqslant 7$, under the assumption that γ is odd.

To extend the result in all cases to lattices with γ even, first apply λ_2 to L to convert to a lattice for which γ is odd. It then follows by the results already established that $\gamma \leqslant 8$. This completes case (i).

Case (ii): The lattice L_2 has a splitting of the type $\langle 2a, 4b, 2^{\gamma+1}c \rangle$ for some $a, b, c \in \mathfrak{u}_2$. Repeatedly applying the mapping λ_2 a total of $\gamma+2$ times will result in a lattice $\hat{L}$ for which $\hat{L}_2 \cong \mathbb{P} \perp \langle 4\epsilon \rangle$, with $\epsilon \in \mathfrak{u}_2$ and either $\mathbb{P} \cong \mathbb{H}$ or $\mathbb{P} \cong \mathbb{A}$; so $\hat{L}$ must occur in the list (5.3). Again we treat the case that γ is odd first. Then L_p is split by $\mathbb{H}$ for the odd primes p for which $p^2|dL$, and $L_p \cong \hat{L}_p^2$ for all other odd primes p.

The argument now proceeds by considering the eight distinct isometry classes over $\mathbb{Z}_2$ for $\langle a, 2b\rangle$. For example, suppose that $\langle a, 2b\rangle \cong \langle 1, 2\rangle$ over $\mathbb{Z}_2$. Then $\Sigma_L \subseteq \mathbb{Q}(\sqrt{-2})$, so $\mathrm{ord}_2 n$ must be even for any spinor exception for $\mathrm{gen}\, L$, so $\mu_1(L) = 2$. When $\hat{L}$ is not #1, $4 \in Q(\mathrm{gen}\, L)$ and 4 is not a spinor exception since $\mathrm{ord}_p dL$ is odd for some odd prime p. Hence, $\mu_2(L) \leqslant 4$. If $\hat{L}$ is #1, then $6 \in Q(L)$. Neither of the lattices $\langle 2, 2\rangle$ or $\langle 2, 6\rangle$ is represented by L over $\mathbb{Z}$, since L_2 does not contain any binary 2-modular sublattice; so the leading binary section of L must be $\langle 2, 4\rangle$. By examining the local structures at 2, 5 and 7 for $\hat{L}$ in (5.3), it can be seen that in all cases at least one of 70 and 140 is represented by $\mathrm{gen}\, L$. Moreover, 70 is never a spinor exception, since $\mathrm{ord}_2 70$ is odd, and 140 can be a spinor exception only when $\mathrm{ord}_5 dL$ and $\mathrm{ord}_7 dL$ are both odd; the latter occurs only when $\hat{L}$ is #16. It can be checked directly that $70 \in Q(\mathrm{gen}\, L)$ when $\hat{L}$ is #16. Since neither 70 nor 140 is represented over $\mathbb{Z}$ by $\langle 2, 4\rangle$, we conclude that $\mu_3(L) \leqslant 140$. Thus $dL \leqslant 1120 < 2^{11}$ and it follows that $\gamma \leqslant 5$. Similar arguments for the seven remaining cases for the isometry class of $\langle a, 2b\rangle$ over $\mathbb{Z}_2$ yield that $\gamma \leqslant 7$ in all cases. Extending to lattices with γ even as before then yields the result $\gamma \leqslant 8$ as claimed.

Case (iii): Successive application of the mapping λ_2 will transform the lattice L into a lattice K with

$$K_2 \cong \begin{cases} \langle 2a', 2b', 2^{\gamma+1-\beta}c'\rangle & \text{if } \beta \text{ is even} \\ \langle 2a', 4b', 2^{\gamma+2-\beta}c'\rangle & \text{if } \beta \text{ is odd} \end{cases}$$

for some $a', b', c' \in \mathfrak{u}_2$. Consequently, $\gamma - \beta \leqslant 8$ when β is even, by case (i) above, and $\gamma - \beta + 1 \leqslant 8$ when β is odd, by case (ii) above. Finally, a bound for β can be obtained by considering the four possible cases for a modulo 8. For example, when $a \equiv 3 \pmod 8$, both 38 and 86 are represented by $\mathrm{gen}\, L$, and neither is a spinor exception. Thus $\mu_1(L) \leqslant 38$ and $\mu_2(L) \leqslant 86$; so the discriminant of the leading binary section of L cannot exceed 3268, which is less than 2^{12}. By considering the splitting of L_2, it follows that $\beta \leqslant 9$. Similar arguments show that $\beta \leqslant 9$ also holds for the other possibilities of a modulo 8. The claimed bound for $\mathrm{ord}_2 dL$ now follows. □

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Appendix A

Table A.1 lists representatives from all equivalence classes of regular ternary lattices M for which δM is squarefree. These correspond to the equivalence classes of regular primitive integral ternary quadratic forms of squarefree discriminant determined in [**12**]. For each lattice M, the table lists a reference number, the matrix of M with respect to a reduced basis, the discriminant of M, and the local splittings of M at each of the prime divisors p of dM. In these splittings, $\mathbb{A}$ and $\mathbb{H}$

TABLE A.1.

Lattice	Matrix	dM	Local structure at $p\|dM$	Lattice	Matrix	dM	Local structure at $p\|dM$	Lattice	Matrix	dM	Local structure at $p\|dM$
#1	$\left\langle\begin{matrix}2&1&0\\1&2&1\\0&1&2\end{matrix}\right\rangle$	4	$M_2 \cong \mathbb{A} \perp \langle 12\rangle$	#9	$\left\langle\begin{matrix}2&0&1\\0&2&1\\1&1&6\end{matrix}\right\rangle$	20	$M_2 \cong \mathbb{A} \perp \langle -4\rangle$ $M_5 \cong \langle 1,1\rangle \perp \langle 5\rangle$	#17	$\left\langle\begin{matrix}2&0&1\\1&6&5\\0&5&10\end{matrix}\right\rangle$	60	$M_2 \cong \mathbb{A} \perp \langle 20\rangle$ $M_3 \cong \langle 1,2\rangle \perp \langle 3\rangle$ $M_5 = \langle 1,1\rangle \perp \langle 5\Delta\rangle$
#2	$\left\langle\begin{matrix}2&1&0\\1&2&0\\0&0&2\end{matrix}\right\rangle$	6	$M_2 \cong \mathbb{H} \perp \langle -6\rangle$ $M_3 \cong \langle 1,1\rangle \perp \langle 6\rangle$	#10	$\left\langle\begin{matrix}2&0&0\\0&2&1\\0&1&8\end{matrix}\right\rangle$	30	$M_2 \cong \mathbb{H} \perp \langle 2\rangle$ $M_3 \cong \langle 1,1\rangle \perp \langle 6\rangle$ $M_5 = \langle 1,1\rangle \perp \langle 5\rangle$	#18	$\left\langle\begin{matrix}2&0&1\\0&4&0\\1&0&6\end{matrix}\right\rangle$	44	$M_2 \cong \mathbb{A} \perp \langle 4\rangle$ $M_{11} \cong \langle 1,\Delta\rangle \perp \langle 11\Delta\rangle$
#3	$\left\langle\begin{matrix}2&1&0\\1&2&1\\0&1&4\end{matrix}\right\rangle$	10	$M_2 \cong \mathbb{H} \perp \langle 6\rangle$ $M_5 \cong \langle 1,\Delta\rangle \perp \langle 5\rangle$	#11	$\left\langle\begin{matrix}2&0&1\\0&2&1\\1&1&22\end{matrix}\right\rangle$	84	$M_2 \cong \mathbb{A} \perp \langle -4\rangle$ $M_3 \cong \langle 1,1\rangle \perp \langle 3\rangle$ $M_7 = \langle 1,1\rangle \perp \langle 7\Delta\rangle$	#19	$\left\langle\begin{matrix}2&1&0\\1&6&3\\0&3&10\end{matrix}\right\rangle$	92	$M_2 \cong \mathbb{A} \perp \langle 20\rangle$ $M_{23} \cong \langle 1,\Delta\rangle \perp \langle 23\Delta\rangle$
#4	$\left\langle\begin{matrix}2&1&0\\1&2&0\\0&0&4\end{matrix}\right\rangle$	12	$M_2 \cong \mathbb{A} \perp \langle 4\rangle$ $M_3 \cong \langle 1,\Delta\rangle \perp \langle 6\rangle$	#12	$\left\langle\begin{matrix}2&1&0\\1&4&2\\0&2&4\end{matrix}\right\rangle$	20	$M_2 \cong \mathbb{H} \perp \langle 12\rangle$ $M_5 \cong \langle 1,\Delta\rangle \perp \langle 5\Delta\rangle$	#20	$\left\langle\begin{matrix}2&1&0\\1&10&6\\0&6&12\end{matrix}\right\rangle$	156	$M_2 \cong \mathbb{A} \perp \langle 20\rangle$ $M_3 \cong \langle 1,1\rangle \perp \langle 3\rangle$ $M_{13} = \langle 1,\Delta\rangle \perp \langle 13\rangle$
#5	$\left\langle\begin{matrix}2&1&0\\1&2&1\\0&1&10\end{matrix}\right\rangle$	28	$M_2 \cong \mathbb{A} \perp \langle 20\rangle$ $M_7 \cong \langle 1,\Delta\rangle \perp \langle 7\Delta\rangle$	#13	$\left\langle\begin{matrix}2&1&0\\1&4&1\\0&1&4\end{matrix}\right\rangle$	26	$M_2 \cong \mathbb{H} \perp \langle 6\rangle$ $M_{13} \cong \langle 1,\Delta\rangle \perp \langle 13\rangle$	#21	$\left\langle\begin{matrix}2&0&0\\0&2&1\\0&1&6\end{matrix}\right\rangle$	22	$M_2 \cong \mathbb{H} \perp \langle -6\rangle$ $M_{11} \cong \langle 1,1\rangle \perp \langle 11\rangle$
#6	$\left\langle\begin{matrix}2&1&0\\1&2&0\\0&0&20\end{matrix}\right\rangle$	60	$M_2 \cong \mathbb{A} \perp \langle 20\rangle$ $M_3 \cong \langle 1,1\rangle \perp \langle 6\rangle$ $M_5 = \langle 1,\Delta\rangle \perp \langle 5\Delta\rangle$	#14	$\left\langle\begin{matrix}2&1&0\\1&4&4\\0&4&14\end{matrix}\right\rangle$	66	$M_2 \cong \mathbb{H} \perp \langle -2\rangle$ $M_3 \cong \langle 1,1\rangle \perp \langle 3\rangle$ $M_{11} = \langle 1,\Delta\rangle \perp \langle 11\rangle$	#22	$\left\langle\begin{matrix}2&0&0\\0&4&1\\0&1&4\end{matrix}\right\rangle$	30	$M_2 \cong \mathbb{H} \perp \langle 2\rangle$ $M_3 \cong \langle 1,2\rangle \perp \langle 3\rangle$ $M_5 = \langle 1,\Delta\rangle \perp \langle 5\rangle$
#7	$\left\langle\begin{matrix}2&0&1\\0&2&1\\1&1&4\end{matrix}\right\rangle$	12	$M_2 \cong \mathbb{H} \perp \langle 20\rangle$ $M_3 \cong \langle 1,1\rangle \perp \langle 3\rangle$	#15	$\left\langle\begin{matrix}2&0&1\\0&4&1\\1&1&6\end{matrix}\right\rangle$	42	$M_2 \cong \mathbb{H} \perp \langle 6\rangle$ $M_3 \cong \langle 1,2\rangle \perp \langle 3\rangle$ $M_7 = \langle 1,1\rangle \perp \langle 7\Delta\rangle$	#23	$\left\langle\begin{matrix}2&1&0\\1&4&2\\0&2&6\end{matrix}\right\rangle$	34	$M_2 \cong \mathbb{H} \perp \langle -2\rangle$ $M_{17} \cong \langle 1,\Delta\rangle \perp \langle 17\Delta\rangle$
#8	$\left\langle\begin{matrix}2&0&0\\0&2&1\\0&1&4\end{matrix}\right\rangle$	14	$M_2 \cong \mathbb{H} \perp \langle 2\rangle$ $M_7 \cong \langle 1,1\rangle \perp \langle 7\rangle$	#16	$\left\langle\begin{matrix}2&0&1\\0&4&0\\1&0&18\end{matrix}\right\rangle$	140	$M_2 \cong \mathbb{A} \perp \langle 4\rangle$ $M_5 \cong \langle 1,\Delta\rangle \perp \langle 5\Delta\rangle$ $M_7 = \langle 1,1\rangle \perp \langle 7\Delta\rangle$	#24	$\left\langle\begin{matrix}2&1&0\\1&4&0\\0&0&6\end{matrix}\right\rangle$	42	$M_2 \cong \mathbb{H} \perp \langle 6\rangle$ $M_3 \cong \langle 1,1\rangle \perp \langle 3\rangle$ $M_7 = \langle 1,\Delta\rangle \perp \langle 7\rangle$

are used to represent the binary even unimodular lattices with matrices

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively, over $\mathbb{Z}_2$, and the symbol Δ denotes a nonsquare unit modulo the designated prime.

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