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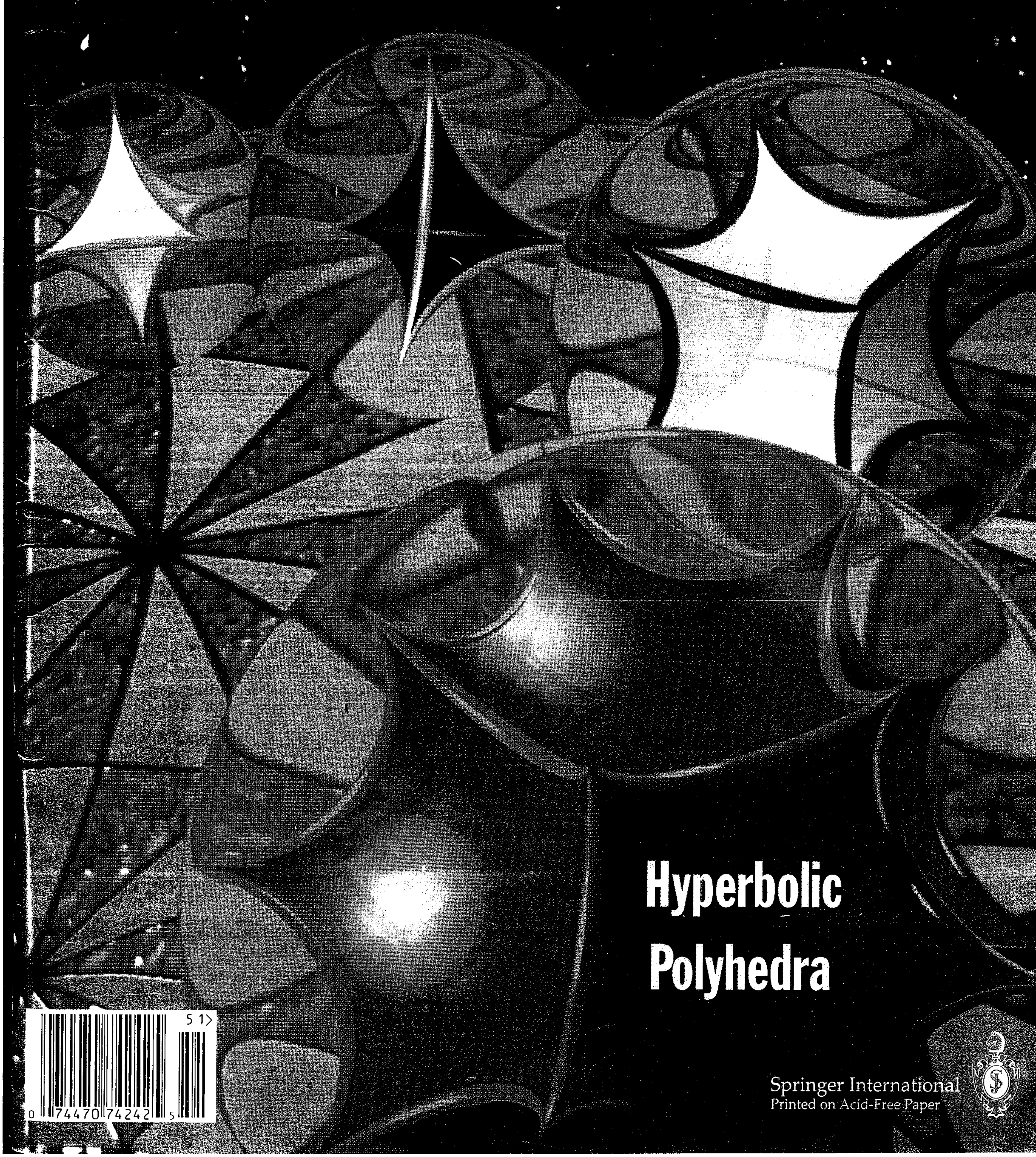

Hyperbolic Polyhedra

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Contents

Departments

Articles

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On the Cover:

The cover may help us all get habituated to the regularities and beauties of hyperbolic space. Thanks to Dr. Konrad Polthier, who works at the Technical University of Berlin and is head of the Computer Graphics group of the Sonderforschungsbereich 288, for the image. This non-Euclidean way of life is explored in the articles of Ruth Kellerhals (pp. 21–30) and William C. Jagy (pp. 31–36).

Squaring Circles in the Hyperbolic Plane

William C. Jagy

The syndicated newspaper column of Marilyn vos Savant was particularly interesting one Sunday in November 1993 [Sa]. Ms. vos Savant announced there that she had no faith whatsoever in the work of Andrew Wiles on Fermat's Last Theorem. In stating her objections to the methodology of Wiles, she wrote that János Bolyai "managed to 'square the circle'—but only by using his own hyperbolic geometry." The word "using" creates the misleading impression that Bolyai used illicit methods to square the circle in the Euclidean plane. What Bolyai did, in fact, was to construct, using the correct intrinsic versions of the compass and straightedge, a square and a circle *in* the hyperbolic plane with the same area. In this article, I will exhibit all possible such examples (Theorem A). I will also show that the square and circle must be constructed simultaneously: there cannot be a construction that begins with a circle of radius r and produces the correct corner angle σ for the square of equal area (Example B); neither can there be a construction beginning with σ that produces the correct r (Example C). Theorem A, discovered independently by the present author, is contained in a 1948 article of Nestorovich [Ne1] that has received little attention in English-language publications. That article also has an example similar to those in Example B, but Example C is not considered there. It may be, therefore, that Example C and the interpretation provided by Theorems B and C are new.

Introduction

With the normalization we will be using, the area of a triangle in $\mathbb{H}^2$ is the same as its "defect," that being π minus the sum of the three angles (the sum is guaranteed smaller than π). It follows that the areas of squares in the hyperbolic plane are bounded above, although the areas of circles are not. Indeed, a convex polygon with n sides has area bounded by $(n-2)\pi$, and this bound is achieved only by figures with sides of infinite length. In contrast, the circle of radius r has area $2\pi(\cosh r - 1)$, and this can assume any positive value. Whether we consider constructibility or not, only circles with area $\leq 2\pi$ (so that $\cosh r \leq 2$) have a companion square with equal area.

The reader may well be more familiar with the theorems of geometry in $\mathbb{H}^2$ than with straightedge-and-compass constructions there. The very simplest constructions used in the Euclidean plane $\mathbb{E}^2$ are also available in the hyperbolic plane, essentially unchanged. If we imagine a creature in $\mathbb{H}^3$ drawing on a flat sheet of paper, our creature can bisect segments, bisect angles, add or subtract segment lengths, add or subtract angles, and draw perpendiculars to lines, either through a point on the line or "dropped" from a point off the line. Differences from $\mathbb{E}^2$ begin with the lack of a unique "parallel" to a given line through a given point. For example, it is not generally possible to *trisect* a line segment [Ma, p. 483].

Rather than "parallel" lines, given a line l and a point P off l, we may always construct rays m_1 and m_2 (half-

William C. Jagy

William C. Jagy usually works on minimal and constant-mean-curvature submanifolds, in situations where a foliation of the submanifold may be specified. He got his doctorate at UC Berkeley and is back there now as visiting scholar, having taught several years at Midwestern State University (Texas) in the interim. His spare time is devoted to soccer (aka football), spending too much money in bookstores, and searching for more permanent employment.

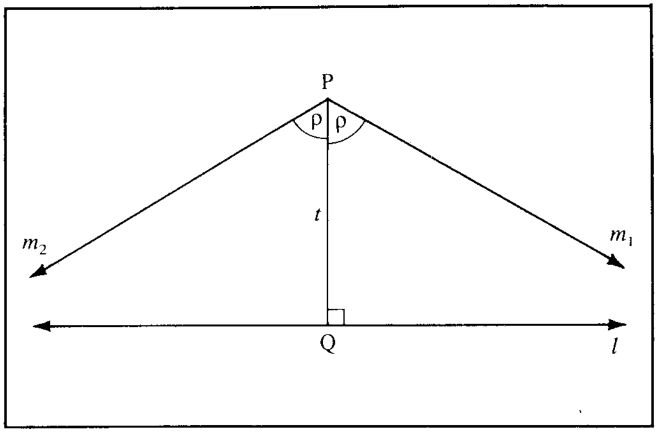

Figure 1.

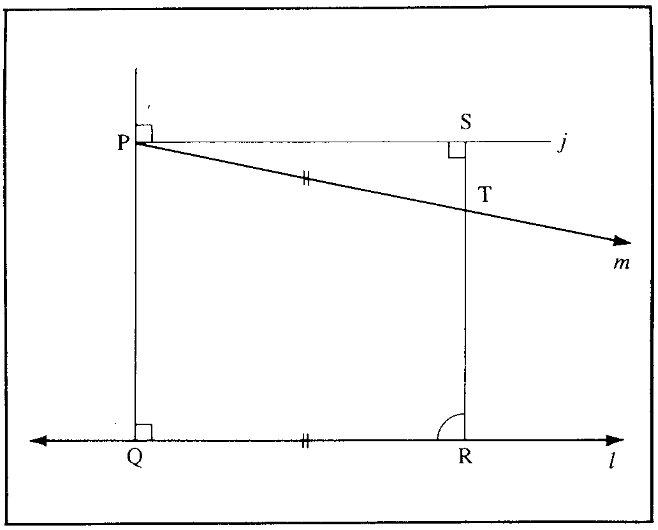

Figure 2.

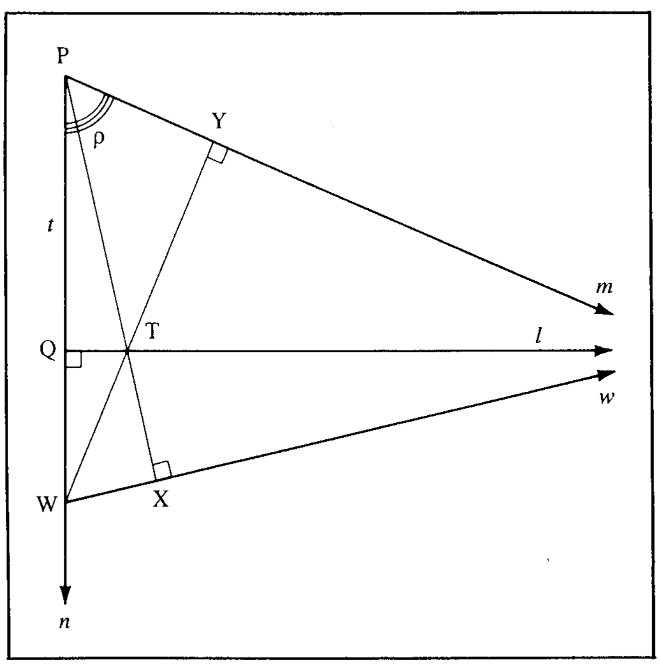

Figure 3.

lines), beginning at P, that are "asymptotic" to l, one in each direction (see Fig. 1). These asymptotic rays, which do not intersect l, can be characterized in several ways. First, if we drop the perpendicular from P to a point Q on l, then the angles between ray m_j ($j = 1,2$) and segment PQ are acute (and equal), and any ray through P that makes a smaller angle with segment PQ must of necessity intersect line l. It is also true that each ray gets "closer and closer" to the line l: for instance, the distance between a point R on m_1 and a point S on l, both at distance s from the fixed point P, goes to 0 as $s \to \infty$.

The concept of asymptotic rays leads to the definition of the function Π, which describes a bijection between nonzero lengths t and acute angles ρ, as illustrated in Figure 1. Then ρ is called the "angle of parallelism" for the length t, and t could be called the "length of parallelism" for ρ. The angle must be measured in radians, which are definable in $\mathbb{H}^2$, based on the assignment of $\pi/2$ radians to the right angle. This defines the monotone decreasing function Π: for this figure, $\Pi(t) = \rho$ and $\Pi^{-1}(\rho) = t$.

János Bolyai provided a construction for the asymptotic ray to a line l in a given direction, beginning at a point P off the line (see Fig. 2). We first drop a perpendicular from P to l, arriving at point Q. Next, draw the ray j through P that is perpendicular to segment PQ in the desired direction. Along the line l, in the same direction from Q, choose any point R, and then drop the perpendicular from R back to j, arriving at S. Use the compass to draw a circle around point P with radius equal to the length of the segment QR. The point of intersection of this circle with segment RS, labeled T, allows us to draw the ray $m = \overrightarrow{PT}$, and this ray is asymptotic to l.

It is also necessary to know how to reverse the previous construction: Given an acute angle ρ, defined by rays m and n through a common point P, construct the ray l that is perpendicular to n and asymptotic to m (see Fig. 3). The illustrated construction is due to Bonola [B, p. 106]. There is the necessity, in this method, of somehow knowing a point W on n that is so far away from P that the ray w through W asymptotic to m makes an acute angle with the segment PW. It is not clear ahead of time how to do this, unless we've already solved the problem for an even smaller angle than ρ. Knowing some point W sufficiently far away, we draw the ray w, and then drop perpendiculars from P and W to X and Y on rays w and m. The segments PX and WY will intersect "in the interior" of the infinite triangle formed by segment PW and rays m and w, at a point we label T. Finally, drop the perpendicular from T to the point Q along segment PW. Extending the segment QT to the ray $l = \overrightarrow{QT}$ gives us the required ray. Bolyai's original work (described in [B, appendix III, pp. 216–226]) gives a lengthy sequence of intermediate constructions to solve the previous problem. Bonola's construction has the virtue of needing no explicit trigonometry for its justification. Martin found a method requiring few steps and no knowledge of any point "sufficiently far away" [Ma, p. 484].

There is an "absolute measurement" of length in $\mathbb{H}^2$, an idea apparently due to Lambert [B, pp. 44–49]. One may associate to a given segment the angle at a vertex of the equilateral triangle with edges congruent to that segment, or some prescribed function of that angle. Further, there are "natural units" of length. One such is Schweikart's constant p, defined by the equation $\Pi(p) = \pi/4$. Another is Gauss's constant k, which can be associated with a relationship among the curves called "horocycles" [Ma, pp. 413–415]. The significance of k in hyperbolic trigonometry is analogous to that of the radius of the sphere in spherical trigonometry. Martin refers to k as the "distance scale." He proves results involving triangles, length, or area for the case $k = 1$, and then describes the adjustment for other values of k, essentially dividing any length by k [Ma, pp. 433–434]. For instance, the comparison between the two constants is $\sinh(p/k) = 1$. I will also restrict to the case $k = 1$. It should be noted that it will not be possible to construct the length we are now calling 1 with compass and straightedge; life is like that.

A short glossary of trigonometry in $\mathbb{H}^2$ is appropriate. This material, with different choices of symbols, is presented in Chapter 32 of [Ma], especially pp. 425 to 433. If we have any triangle (Fig. 4), we find a Law of Sines, along with two distinct versions of the Law of Cosines. The extra law can be said to result from the fact that any two triangles in $\mathbb{H}^2$ that are similar are necessarily congruent.

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c},$$

$$\cos\alpha = \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c},$$

$$\cosh a = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}.$$

For a triangle with a right angle at vertex C, that is, $\gamma = \pi/2$, we find a version of the Pythagorean Theorem and various other facts:

$$\cosh c = \cosh a \cosh b, \qquad \cosh c = \cot\alpha\cot\beta,$$

$$\cosh a = \frac{\cos\alpha}{\sin\beta}, \qquad \sin\alpha = \frac{\sinh a}{\sinh c},$$

$$\cos\alpha = \frac{\tanh b}{\tanh c}, \qquad \tan\alpha = \frac{\tanh a}{\sinh b}.$$

We have already seen the construction of asymptotic rays, resulting in a type of infinite, "singly asymptotic" right triangle. The trigonometry for these reduces to relations between the finite edge, of length t, and the acute angle $\Pi(t)$:

$$\tanh t = \cos\Pi(t), \qquad \sinh t = \cot\Pi(t),$$

$$\cosh t = \csc\Pi(t), \qquad e^{-t} = \tan\frac{\Pi(t)}{2}$$

Other Constructions

In discussing the problem of "squaring the circle," Bolyai introduced an angle, which I shall call θ, associated with the radius r of a circle. It should be noted (having arranged $k = 1$) that the area within a circle in $\mathbb{H}^2$ is expressed by $4\pi\sinh^2(r/2)$. The angle θ will be constructed so that $\tan\theta = 2\sinh(r/2)$. The result is that the area of the circle is equal to $\pi\tan^2\theta$. There are explicit methods for beginning with θ and constructing r [Ma, p. 489] and for beginning with r and constructing θ. As a result, questions of constructibility for r can be rewritten as questions about θ. We exhibit the standard diagram (Fig. 5) illustrating the relationship between θ and r, which is used in both these constructions.

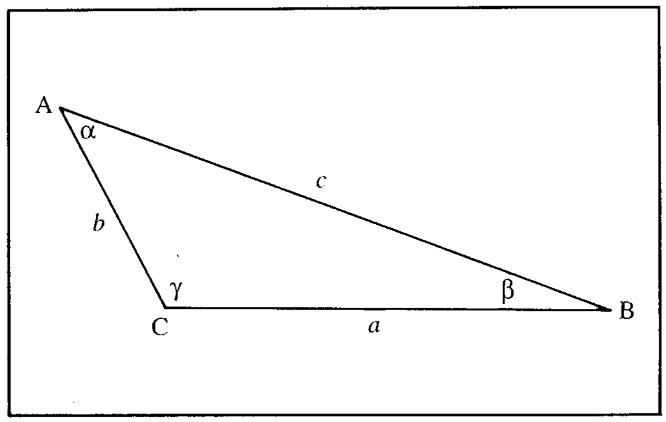

Figure 4.

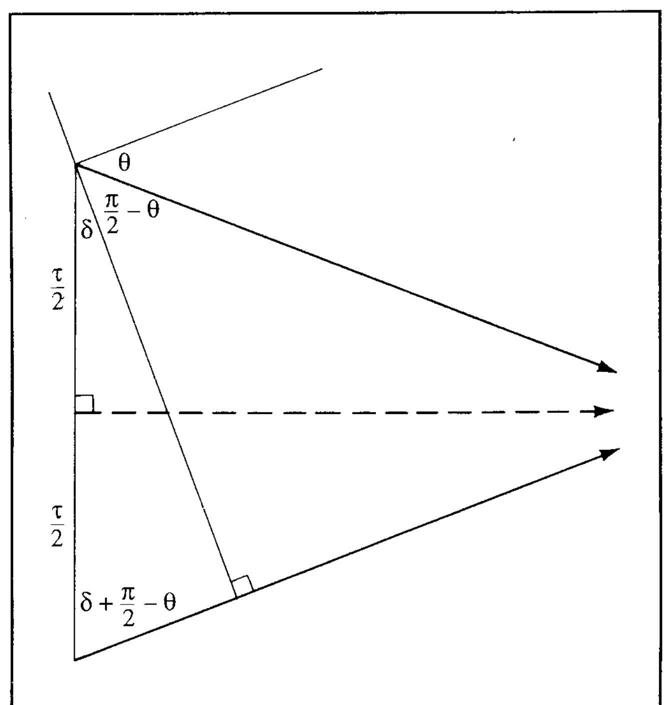

Figure 5.

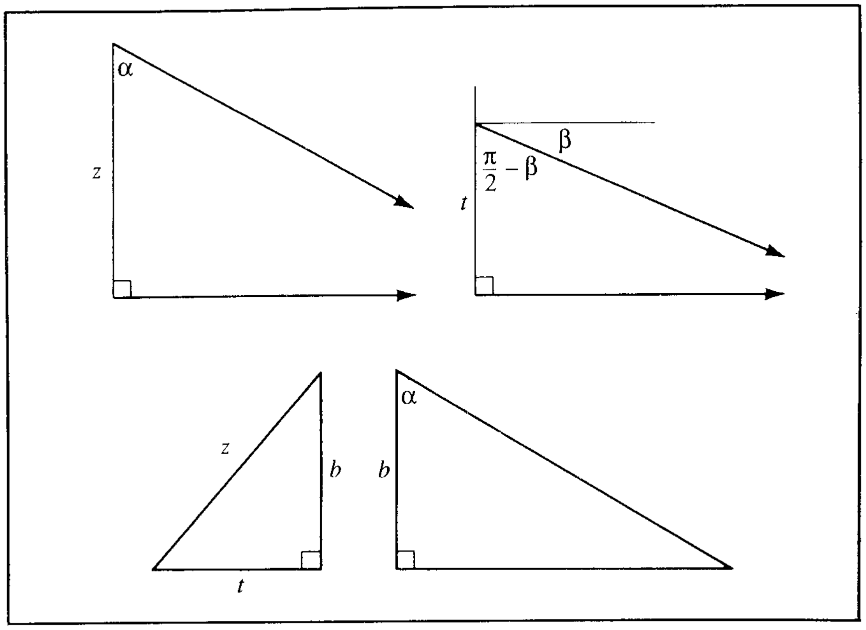

Figure 6.

It is necessary to describe the construction of a right triangle with the other two angles α and β prescribed, and with $\alpha + \beta < \pi/2$ (see Fig. 6). First, construct lengths z and t so that $\Pi(z) = \alpha$ and $\Pi(t) = \pi/2 - \beta$. As $\alpha < \pi/2 - \beta$, it follows that $z > t$, so we may use the compass to draw a right triangle with hypotenuse z and one leg t. Call the other leg of this right triangle length b. Finally, construct the right triangle with one leg of length b and the adjacent angle equal to α. A combination of the various trigonometric relations shows that the angle opposite to the edge of length b is, in fact, equal to β.

Comparison of Constructibility in $\mathbb{E}^2$ and $\mathbb{H}^2$

Our main tool is an observation that an angle is constructible in $\mathbb{H}^2$ if and only if it is constructible in $\mathbb{E}^2$ [Ma, p. 483]. This follows from comparing trigonometry in $\mathbb{H}^2$ and $\mathbb{E}^2$, as explained below. It is uncertain where the observation was first recorded.

Suppose we give the name E to the set of lengths in $\mathbb{E}^2$ that are constructible, beginning with some assigned length denoted 1. The elements of E are thought of as real numbers. By courtesy, the length 0 is a member of E, and if $s < 0$ and $|s| \in E$, we agree to say $s \in E$. Recall that one can use the compass and straightedge in $\mathbb{E}^2$ to add or subtract lengths, multiply or divide them, and produce the square root of a given length. By considering intersections of lines and circles, it is shown that the preceding operations characterize E exactly: it is a field and is the smallest subfield of $\mathbb{R}$ that is "closed under square roots": if $s \in E$, $s > 0$, then $\sqrt{s} \in E$.

As to the hyperbolic plane, it turns out that a length t in $\mathbb{H}^2$ is constructible with compass and straightedge if and only if $\sinh t \in E$, or $\cosh t \in E$, or $\tanh t \in E$, these conditions being equivalent. Again, we consider 0 constructible, as also negative t when $|t|$ is constructible. Knowing the correct result, it is not difficult to prove this theorem. The original proof of the theorem is spread over at least four articles. These begin with two by D. D. Mordukhaĭ-Boltovskoĭ ([M-B1] and [M-B2]), which together show both sides of the if-and-only-if statement, but allow the use of a third drawing instrument called the "hypercompass." Later articles by Nestorovich established that all constructions that include the extra "hypercompass" can be performed with just the compass and straightedge. The interested reader may consult the monograph of Smogorzhevskiĭ [Sm] or the problem book of Nestorovich [Ne2]. Another proof, discovered without knowledge of [M-B2], appears in the book of Kagan [Ka].

By considering a right triangle with any side of length 1 we find that an angle α is constructible in $\mathbb{E}^2$ if and only if $\sin\alpha$, or $\cos\alpha$, or $\tan\alpha$ is in E, these conditions being equivalent. Consider a right triangle in $\mathbb{H}^2$ with one side equal to Schweikart's length p, which satisfies $\sinh p = 1$. Using hyperbolic trigonometry, we find that an angle η is constructible in $\mathbb{H}^2$ if and only if $\sin\eta$, or $\cos\eta$, or $\tan\eta$ is in E, these conditions being equivalent, i.e.,

An angle α can be constructed in $\mathbb{H}^2$ if and only if it can be constructed in $\mathbb{E}^2$.

Matching Areas in $\mathbb{H}^2$

We have already mentioned the auxiliary angle θ, with the property that the area of the circle of radius r is $\pi \tan^2\theta$. A "square" will be a convex quadrilateral with four equal edges and four equal angles (which must be acute). Let us refer to the corner angle of the square as σ. This square can be constructed from eight right triangles with angles $\sigma/2$ and $\pi/4$ (Fig. 7). In each triangle, denote by $y/2$ the length of the side opposite the angle $\pi/4$, so that the edge of the resulting square is length y. One of the trigonometric relations reads $\cosh(y/2) = \cos(\pi/4) \div \sin(\sigma/2)$, from which follows the remark (for the square and circle of equal area) that $\cosh y = \tan^2\{(\pi \cosh r)/4\}$. Since the area of each triangle is its defect, $\pi - (\pi/2 + \pi/4 + \sigma/2) = \pi/4 - \sigma/2$, the area of the square is $8(\pi/4 - \sigma/2) = 2\pi - 4\sigma$.

Our matching area problem can now be written in terms of angles in $\mathbb{H}^2$, that is, $2\pi - 4\sigma = \pi \tan^2\theta$. The conditions that the square and circle be constructible are therefore expressible in terms of the constructibility—in $\mathbb{E}^2$—of angles σ and θ that satisfy $2\pi - 4\sigma = \pi \tan^2\theta$.

Suppose we give the symbol ω to the common area of the square and circle. Since $\omega = 2\pi - 4\sigma$, it follows that ω is a constructible angle in $\mathbb{E}^2$. We are considering $\omega = \pi \tan^2\theta$. As θ is a constructible angle, $\tan\theta$ and its square are elements of E. If we write $x = \tan^2\theta$, we have a constructible angle ω and a constructible length x in $\mathbb{E}^2$ such that $\omega = \pi x$. To relate the various symbols, we record

$$\omega = 2\pi - 4\sigma = \pi x = \pi \tan^2\theta = 4\pi \sinh^2\left(\frac{r}{2}\right)$$
$$= 2\pi(\cosh r - 1).$$

We note that $2\sigma + \pi \cosh r = 2\pi$, as well as $x + 2 = 2\cosh r$.

The equation $\omega = \pi x$ can be analyzed using a famous result about transcendental numbers over $\mathbb{Q}$. For convenience, we refer to those complex numbers that are algebraic over $\mathbb{Q}$ as simply "algebraic" and assign to them the symbol $\mathbb{A}$. Recall that the algebraic numbers $\mathbb{A} \subset \mathbb{C}$ form a field, and that $i \in \mathbb{A}$. One may find the following theorem stated in [Ni, p. 134]:

GELFOND–SCHNEIDER THEOREM (GS): *If φ and χ are nonzero algebraic, $\varphi \neq 1$, and $\chi \notin \mathbb{Q}$, then any value of φ^χ is transcendental.*

Remarks. The value of φ^χ is defined to be $\exp(\chi \log \varphi)$, so it is multivalued like the logarithm. Also, GS applies when χ is an algebraic number with nonzero imaginary part, such as $-2i$. Since one value for i^{-2i} is e^π, this shows that e^π is transcendental. Finally, GS prohibits $\varphi = 1$, but allows $\varphi = -1$.

The reader will note that $E \subset \mathbb{A}$, so that $E(i)$ is also a subfield of $\mathbb{A}$. We may profitably return to the equation $\omega = \pi x$, between a constructible angle ω and a constructible length x in $\mathbb{E}^2$. Since ω is constructible, $\sin\omega = \sin\pi x$ and $\cos\omega = \cos\pi x$ are both in E. It follows that $e^{i\pi x} = \cos\pi x + i\sin\pi x$ belongs to $E(i) \subset \mathbb{A}$. If we choose $\log(-1) = \pi i$, this means that $(-1)^x = \exp(x\log(-1)) = \exp(i\pi x)$ is in $\mathbb{A}$. On the other hand, $x \in E \subset \mathbb{A}$, so that $(-1)^x \in \mathbb{A}$ implies x is rational by GS. From a relationship exhibited earlier, $x + 2 = 2\cosh r$, we note that $\cosh r$ must also be rational.

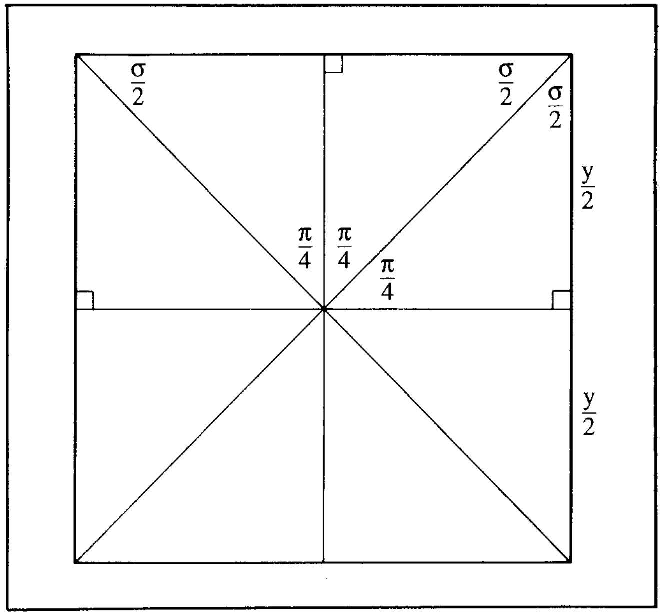

Figure 7.

Now that $x \in \mathbb{Q}$, suppose we write x as m/n in "lowest terms," that is, $m, n \in \mathbb{Z}$, $n \geq 1$, and $\gcd(m, n) = 1$. Then there must be integers u and v such that $um + vn = 1$. If we multiply this through by π/n, we find $um\pi/n + v\pi = \pi/n$, or $u\omega + v\pi = \pi/n$. As u and v are integers, and ω is a constructible angle, π/n must also be a constructible angle. This is related to a famous question, with a famous answer, supplied by Gauss and Wentzel (see [Kl]). By placing an angle $2\pi/n$ at the center of a circle and copying it n times, we construct, in $\mathbb{E}^2$, a regular polygon of n sides. The famous answer implies that n must have prime factorization $n = 2^j F_{i_1} F_{i_2} \cdots F_{i_r}$ (where the F_{i_h} are *distinct* primes of the form $1 + 2^{2^l}$), $j \geq 0$, $r \geq 0$. The F_{i_h} are often called "Fermat numbers," and only five are known to be prime: writing $F_l = 1 + 2^{2^l}$, these are $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65,537$. The next one, F_5, has a factor of 641, and F_6 has a factor of 274,177. In the year 1987, the values F_5 through F_{21} were known to be composite [R, pp. 71–74]. In this year 1994, when secret codes are based on the difficulty today's computers have factoring a "random" number with 200 digits, it is sobering to note that even F_{10} has over 300 digits in decimal notation.

Returning to the equations $\omega = 2\pi - 4\sigma$ and $\omega = \pi \tan^2\theta = \pi x$, we find that σ, the corner angle of the square, satisfies $\sigma = (2\pi - \omega)/4$, so that σ is a rational multiple of π, with denominator n as described above. We shall reject the "square" with $\sigma = \pi/2$; if that made any sense in $\mathbb{H}^2$, it would be a single point, with area 0. We shall allow the square with four infinite edges and $\sigma = 0$, of area 2π. There is a countably infinite set of satisfactory angles σ, and they are dense in the interval from 0 to $\pi/2$. We have shown that the problem of constructing the two figures is equivalent to the problem of constructing regular polygons in $\mathbb{E}^2$ (or, indeed, in $\mathbb{H}^2$):

THEOREM A. *Suppose a square of corner angle σ and a circle of radius r in $\mathbb{H}^2$ have the same area ω, so that $\omega \leq 2\pi$. Then both are constructible if and only if σ satisfies these conditions: $0 \leq \sigma < \pi/2$, and σ is an integer multiple of $2\pi/n$, n a positive integer such that the regular polygon of n sides can be constructed with compass and straightedge in $\mathbb{E}^2$.*

Note that we simultaneously produced the square and the circle from auxiliary information, and required that both be constructible with compass and straightedge. The natural question occurs, "What of a method that begins with any σ or r, and produces the other?" Theorem A is silent on this question; however, there are no such methods. We exhibit two (dense sets of) examples, showing that there is no general method in either direction.

Example B. Let m/n be a rational number in lowest terms, such that n is not a power of 2, but has some odd prime factor d. Then $\theta = \arctan(m/n)$ is a constructible angle. But $\omega = \pi \tan^2\theta = \pi m^2/n^2$ cannot be constructible, as that would imply constructibility for the regular polygon of d^2 sides. *If there were* a construction that began with r (whence θ) and produced the correct ω (whence σ), then whenever r was constructible, the resulting σ would be the outcome of a long construction. Our family of examples provides constructible r with the corresponding σ nonconstructible, thus precluding the existence of such a method.

THEOREM B. *There can be no general construction in $\mathbb{H}^2$ that begins with the radius r of a circle and produces the corner angle σ of the square with matching area.*

Remark. The article [Ne1] provides the example

$$\sinh(r/2) = \frac{1}{2}\sqrt{2-\sqrt{2}}.$$

This means that $\theta = \arctan\sqrt{2-\sqrt{2}}$, so r and θ are constructible. For the corresponding square, however, $\sigma = \pi\sqrt{2}/4$, which is not constructible. The conclusion reached from this example translates as: *The class of circulable squares is wider than the class of quadrable circles.*

For the next example, not contemplated in [Ne1], we quote another theorem [Ni, p. 41]:

OLMSTED'S THEOREM. *If τ is a rational multiple of π, the only possible values of $\tan\tau$ that are rational are 0, 1, and -1.*

Example C. Let q be some rational number, $q > 0$, $q \neq 1$. As $q \in E$, we can certainly construct the angle $\sigma = \arctan q$. Since $\sigma \neq \pi/4$, Olmsted's theorem shows that σ/π is irrational. As $\cos\sigma = 1/\sqrt{1+q^2} \in E$ and $\sin\sigma = q/\sqrt{1+q^2} \in E$, then $e^{i\sigma} \in E(i) \subset \mathbb{A}$. Choose $\log(-1) = \pi i$, so $(-1)^{\sigma/\pi} = \exp((\sigma/\pi)\log(-1)) = \exp(\sigma i)$. This time, since $(-1)^{\sigma/\pi}$ is algebraic, we use GS to conclude that σ/π is transcendental. Since $\omega = 2\pi - 4\sigma$, it follows that ω/π is transcendental. As $\omega = \pi\tan^2\theta$, we have $\omega/\pi = \tan^2\theta$, and so we know that $\tan^2\theta$ is transcendental, finally showing that $\tan\theta$ is itself transcendental. Because $E \subset \mathbb{A}$, this means that, although $\sigma = \arctan q$ is constructible, the angle θ appropriate to σ is not. *If there were* a construction that began with σ (whence ω) and produced the correct θ (whence r), then whenever we constructed σ, the resulting r would have been constructed. Our family of examples precludes the existence of such a method.

THEOREM C. *There can be no general construction in $\mathbb{H}^2$ that begins with the corner angle σ of a square and produces the radius r of a circle with matching area.*

References

[B] Roberto Bonola, *Non-Euclidean Geometry*, New York: Dover (1955).

[C] H. S. M. Coxeter, *Non-Euclidean Geometry*, 5th ed., Toronto: University of Toronto Press (1955).

[G] Marvin J. Greenberg, *Euclidean and Non-Euclidean Geometries*, 3rd ed., New York: W. H. Freeman (1993).

[Ka] V. F. Kagan, *Foundations of Geometry, Part I*, Moskva: Gosudarstvennoe Izdatel'stvo Tekhniko-Teoreticheskoĭ Literatury (1949). [In Russian.] [Reviewed in MR 12, 731–732.]

[Kl] Israel Kleiner, *Mathematical Intelligencer* 15 (1993), no. 3, pp. 73–75. [Book review.]

[Ma] George E. Martin, *The Foundations of Geometry and the Non-Euclidean Plane*, New York: Springer-Verlag (1986).

[M-B1] D. D. Mordukhaĭ-Boltovskoĭ, On geometric constructions in Lobachevskiĭ space, in *In Memoriam Lobatschevskii*, Glavnauka (1927), Vol. 2, pp. 67–82. [In Russian.]

[M-B2] D. D. Mordukhaĭ-Boltovskoĭ, On constructions using algebraic curves in Euclidean and non-Euclidean spaces, *Zh. Matematichnogo Tsiklu* 1(3) (1934), 15–30. [In Ukrainian.]

[Ne1] N. M. Nestorovich, On the quadrature of the circle and circulature of a square in Lobachevskiĭ space (Russian). *Dokl. Akad. Nauk SSSR (N.S.)* 63 (1948), 613–614. [In Russian.] [Reviewed in MR 10, 562.]

[Ne2] N. M. Nestorovich, *Geometric Constructions in the Lobachevskiĭ Plane*, Moskva: Gosudarstvennoe Izdatel'stvo Tekhniko-Teoreticheskoĭ Literatury (1951). [In Russian.] [Reviewed in MR 13, 969–970.]

[Ni] Ivan Niven, *Irrational Numbers*, Washington, DC: Mathematical Association of America (1956).

[R] Paulo Ribenboim, *The Book of Prime Number Records*, New York: Springer-Verlag (1988).

[Sa] Marilyn vos Savant, "Ask Marilyn," *Parade Magazine*, 21 November 1993.

[Sm] A. S. Smogorzhevskiĭ, *Geometric Constructions in the Lobachevskiĭ Plane*, Moskva: Gosudarstvennoe Izdatel'stvo Tekhniko-Teoreticheskoĭ Literatury (1951). [In Russian.] [Reviewed in MR 14, 575.]

Department of Mathematics
University of California
Berkeley, CA 94720
USA