List of $f(x,y)+z^3+Az^2+Bz$

William C. Jagy

April 2, 2009

1 Easy

$$-108: \quad 4x^2+2xy+7y^2+z^3$$

$$-243: \quad 7x^2+3xy+9y^2+z^3$$

2 Hard but with $\pm$ symmetry

$$-23: \quad 2x^2+xy+3y^2+z^3-z$$

$$-31: \quad 2x^2+xy+4y^2+z^3+z$$

$$-59: \quad 3x^2+xy+5y^2+z^3+2z$$

$$-76: \quad 4x^2+2xy+5y^2+z^3-2z$$

$$-92: \quad 3x^2+2xy+8y^2+z^3-z$$

$$-124: \quad 5x^2+4xy+7y^2+z^3+z$$

$$-211: \quad 5x^2+3xy+11y^2+z^3-2z$$

$$-283: \quad 7x^2+5xy+11y^2+z^3+4z$$

$$-499: \quad 5x^2+xy+25y^2+z^3+4z$$

$$-643: \quad 7x^2+xy+23y^2+z^3-2z$$

$$-652: \quad 4x^2+2xy+41y^2+z^3-8z$$

Note that in $D=652$ of Table 1 on page 134 of Hudson and Williams [2] I take a translate of x^3+3x^2-5x+3 and use $x^3-8x+10$.

3 Hard, no symmetry

For these I prefer to change the $\pm$ sign for the z^2 term, so that the unrepresented number with smallest absolute value is the constant term of Table 1 on page 134 of Hudson and Williams [2].

$$-44: \quad 3x^2 + 2xy + 4y^2 + z^3 - z^2 - z$$

$$-83: \quad 3x^2 + xy + 7y^2 + z^3 - z^2 + z$$

$$-107: \quad 3x^2 + xy + 9y^2 + z^3 - z^2 + 3z$$

$$-139: \quad 5x^2 + xy + 7y^2 + z^3 + z^2 + z$$

$$-172: \quad 4x^2 + 2xy + 11y^2 + z^3 + z^2 - z$$

$$-268: \quad 4x^2 + 2xy + 17y^2 + z^3 - 2z^2 - 2z$$

$$-307: \quad 7x^2 + xy + 11y^2 + z^3 + z^2 + 3z$$

$$-331: \quad 5x^2 + 3xy + 17y^2 + z^3 + 2z^2 + 4z$$

$$-379: \quad 5x^2 + xy + 19y^2 + z^3 - z^2 + z$$

$$-547: \quad 11x^2 + 5xy + 13y^2 + z^3 - z^2 - 3z$$

$$-883: \quad 13x^2 + xy + 17y^2 + z^3 - 5z^2 - 5z$$

$$-907: \quad 13x^2 + 9xy + 19y^2 + z^3 - 5z^2 + z$$

4 Sample conjectures

-23: **We Conjecture** that $2x^2 + xy + 3y^2 + z^3 - z \neq \pm C$ whenever $C > 0$ is an integer prime to 2 and 3 and there is an integer F with

$$27C^2 - 23F^2 = 4.$$

-44: **We Conjecture** that $3x^2 + 2xy + 4y^2 + z^3 - z^2 - z \neq C$ whenever C is an integer and there is an **odd** integer F with

$$27C^2 + 22C - 5 = 44F^2,$$

or

$$(27C + 11)^2 - 1188F^2 = 256.$$

The first few C values (in absolute value) are $\{1, -6359, -130439, 618042481\}$. Except for the (C, F) pair $(1, -1)$ we take C and F with the same sign in the recipe for producing new (C, F) pairs:

$$\begin{pmatrix} -48599 & -62040 \\ -38070 & -48599 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -19800 \\ -15510 \end{pmatrix} = \begin{pmatrix} -6359 \\ -4981 \end{pmatrix}$$

$$\begin{pmatrix} -48599 & -62040 \\ -38070 & -48599 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -19800 \\ -15510 \end{pmatrix} = \begin{pmatrix} -130439 \\ -102179 \end{pmatrix}$$

$$\begin{pmatrix} -48599 & -62040 \\ -38070 & -48599 \end{pmatrix} \begin{pmatrix} -6359 \\ -4981 \end{pmatrix} + \begin{pmatrix} -19800 \\ -15510 \end{pmatrix} = \begin{pmatrix} 618042481 \\ 484143239 \end{pmatrix}$$

$$\begin{pmatrix} -48599 & -62040 \\ -38070 & -48599 \end{pmatrix} \begin{pmatrix} -130439 \\ -102179 \end{pmatrix} + \begin{pmatrix} -19800 \\ -15510 \end{pmatrix} = \begin{pmatrix} 12678370321 \\ 9931594441 \end{pmatrix}$$

Even when there is no $\pm$ symmetry you can see that there is approximate exponential growth in $|C|$. Any hope of proving these uses the methods of Spearman and Williams [3]. I got valuable insights from Buell [1].

References

[1] D. A. Buell. *Binary Quadratic Forms: Classical Theory and Modern Computations*. Springer-Verlag, 1989.

[2] Richard H. Hudson and Kenneth S. Williams. Representation of primes by the principal form of discriminant $-D$ when the classnumber $h(-D)$ is 3. *Acta Arithmetica*, 57:131–153, 1991.

[3] Blair K. Spearman and Kenneth S. Williams. The cubic conguence $x^3 + Ax^2 + Bx + C \equiv 0 \bmod p$ and binary quadratic forms. *Journal of the London Mathematical Society*, 46:397–410, 1992.