FRACTIONAL ITERATION NEAR A FIX POINT OF MULTIPLIER 1

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1. *Introduction*

An analytic function $f(z)$ is said to have a fix point $\zeta \neq \infty$ of multiplier 1, if $f(\zeta) = \zeta, f'(\zeta) = 1$. The function then has an expansion,

$$f(z) = \zeta + (z-\zeta) + \sum_{n=m+1}^{\infty} a_n (z-\zeta)^n,\ a_{m+1} \neq 0,\ m \geqslant 1. \tag{1}$$

It has been shown (e.g. Baker [1]) that there is, for every complex λ, a unique formal iterate,

$$f_\lambda(z) = \zeta + (z-\zeta) + \sum_{n=m+1}^{\infty} a_n(\lambda)(z-\zeta)^n,\ a_{m+1}(\lambda) = \lambda a_{m+1}, \tag{2}$$

where the $a_n(\lambda)$ are well defined polynomials in λ determined by comparing coefficients in the formal identity $f_\lambda \circ f(z) = f \circ f_\lambda(z)$. For positive integral values $\lambda = n$ the series (2) is the same as that of the nth iterate of $f(z) = f_1(z)$ and by analogy the $f_\lambda(z)$ are in general called the fractional iterates.

Without loss of generality, we choose our fix point at the origin. For simplicity, we shall work with the case $m = 1$ when (1) and (2) reduce to

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ a_2 \neq 0, \tag{3}$$

$$f(z) = z + \sum_{n=2}^{\infty} a_n(\lambda) z^n,\ a_2(\lambda) = \lambda a_2, \tag{4}$$

respectively.

We note that the series (4) does not necessarily have a positive radius of convergence for each λ; in fact it is shown in [1] that the values of λ corresponding to a positive radius of convergence either fill out the whole complex plane, or form a discrete one- or two-dimensional lattice. When the values fill out the whole plane we shall call f *embeddable*.

In [2] (*cf.* also [3] and [7]) it was shown that if $f(z)$ in (3) is meromorphic in the plane, then it is not embeddable except in the case $f(z) = z/(1-a_2 z)$. The question was raised as to whether this result extends to any f which is single-valued. Example 1 below shows that this result does not extend in full generality, while Theorem 1 gives a new class of non-embeddable single-valued functions.

THEOREM 1. *Let D be a domain bounded by a finite set of non-intersecting analytic curves, denoted by δ. Suppose also that D is bounded and contains the origin. If the function $f(z)$ is regular and single-valued in D, with an expansion of the form* (3) *at the origin, and if the curves δ form a natural boundary for f, then f is not embeddable.*

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Thus, for example,

$$f(z) = \sum_{n=0}^{\infty} x^{2^n},$$

which has the unit disc as a natural boundary, is not embeddable. The assumption that the boundary δ of D is analytic is essential, as is shown by the

EXAMPLE 1. *There exists a non-analytic Jordan curve γ and a function $f(z)$ such that*

(i) *γ lies in the disc $|z| < 1$ and the region D bounded by γ and the circumference $|z| = 1$ contains $z = 0$,*

(ii) *f has an expansion*

$$f(z) = z + \sum_{2}^{\infty} b_n z^n, \; b_2 \neq 0,$$

convergent in $|z| > \rho$ for some $\rho > 0$,

(iii) *D is the exact region of existence of the function f obtained by analytic continuation of the expansion in (ii) and f is single-valued,*

(iv) *f is embeddable.*

One can, however, prove

THEOREM 2. *Let D be a domain bounded by a finite number of non-intersecting Jordan curves. Suppose also that D is bounded and contains the origin. If the function $f(z)$ is regular and single-valued in D, with an expansion of the form* (3) *at the origin, if the curves δ form a natural boundary for D and if the boundary values of f on δ all lie outside $\bar{D}$, then f is non-embeddable.*

A case where D is bounded by a discrete set is given by

THEOREM 3. *If $f(z)$ is single-valued and meromorphic in the whole complex plane except for at most a countable number of essential singularities, each isolated from the rest, and if near the origin $f(z)$ has an expansion of the form* (3), *then $f(z)$ is not embeddable, except in the case $f(z) = z/(1-a_2 z)$.*

Finally we turn our attention to the values of λ which correspond to a positive radius of convergence of $f_\lambda(z)$ and show that the case of the two-dimensional lattice, mentioned above, cannot occur.

THEOREM 4. *If the set of values of λ corresponding to a positive radius of convergence for $f_\lambda(z)$ in* (4) *includes a two-dimensional lattice, then $f(z)$ is embeddable.*

Thus, if f is not embeddable, f_λ converges only for $\lambda = n\lambda_0$, where $\lambda_0 \neq 0$ is some fixed constant and n runs through the integers.

The proofs given here are taken from the author's University of London Ph.D. thesis [6]. Recently J. Écalle [4] has announced in a note that he has found an independent proof of Theorem 4.

2. *Preliminary results and notation*

It is sometimes convenient to transfer the fix point to ∞. We shall employ the substitutions $z = k/t$, $z_1 = k/t_1$, choosing $a_2 k = -1$. Applying these to the transformation $z_1 = f(z)$ of (3) we get

$$t_1 = t+1+\sum_{n=1}^{\infty} b_n t^{-n} = g(t) \tag{5}$$

with fix point at ∞ and with b_1 satisfying

$$b_1 = (a_2{}^2 - a_3)/a_2{}^2 . \tag{6}$$

Similarly, (4) transforms to give

$$t_\lambda = t+\lambda+\sum_{n=1}^{\infty} b_n(\lambda)\, t^{-n} = g_\lambda(t) \tag{7}$$

where $g_\lambda(t)$ form the unique family of formal series (7) commuting with (5). The series (4) is embeddable precisely when the series (7) converges for some $t \neq \infty$, for every λ. We shall from now on assume $g = g_1$ convergent for $|t| < R$.

We quote Lemmas 1–3 from [**1**].

LEMMA 1. [**1**; p. 272]. *If $g(t)$ is as in* (5) *and if $\mathscr{D}(K) = \bigcup_\alpha \mathscr{C}(\alpha, K)$, where the union is taken over all α in $-\pi/4 \leqslant \alpha \leqslant \pi/4$, and where $\mathscr{C}(\alpha, K)$ is the half plane $\{t \mid \operatorname{Re}(te^{-i\alpha}) > K\}$, then for all sufficiently large K $(> R)$, $g_n(t)$ is regular,*

$$g_n(t) \in \mathscr{D}(K), \qquad n = 1, 2, \ldots, \tag{8}$$

and

$$\operatorname{Re} g_n(t) \to \infty \quad as \quad n \to \infty \tag{9}$$

for all t in the closure $\bar{\mathscr{D}}(K)$ of $\mathscr{D}(K)$. Moreover (9) *holds locally uniformly in $\mathscr{D}(K)$.* (*cf.* [**1**; p. 273 (21)]).

LEMMA 2 [**1**; p. 273]. *For all sufficiently large K the domain $\mathscr{D}(K)$ of Lemma* 1 *has the following properties.*

$$A(t) = \lim_{n\to\infty} \{g_n(t) - n - b_1 \log n\} \tag{10}$$

(*where b_1 is as defined in* (6)) *exists uniformly for $t \in \mathscr{D}(K)$. Moreover, $A(t)$ is regular and univalent in $D(K)$ and $A'(t) \to 1$ uniformly as $t \to \infty$ in $\mathscr{D}(K)$. Also*

$$A\{g_n(t)\} = A(t)+n \quad for \quad t \in \mathscr{D}(K). \tag{11}$$

LEMMA 3 [**1**; p. 279]. *Let*

$$f_\lambda(z) = z+\lambda a_{m+1} z^{m+1} + \sum_{n=m+2}^{\infty} a_n z^n,\ a_{m+1} \neq 0,\ m \geqslant 1,$$

be a commuting family of formal power series. For $p > 0$, let Ω_p be the class of complex λ with $|\lambda| \leqslant p$ for which $f_\lambda(z)$ has positive radius of convergence. Then there exist constants $\rho > 0$ and $M > 0$ such that

(i) *$f_\lambda(z)$ converges in $|z| \leqslant \rho$ for all $\lambda \in \Omega_p$*

and

(ii) *$|f_\lambda(z)| < M$ uniformly for all $|z| \leqslant \rho$ and all $\lambda \in \Omega_p$.*

LEMMA 4. (Szekeres [7]). *If the series*

$$t_\lambda = g_{\lambda(t)} = t+\lambda+\sum_{n=1}^{\infty} b_n(\lambda)\, t^{-n}$$

in (7) *has a positive radius of convergence for every* λ, *i.e.* g *in* (5) *is embeddable, then* $D(t) = A'(t)$ *is regular in a full neighbourhood of* $t = \infty$ *and has an expansion*

$$D(t) = 1-\frac{b_1}{t}+\sum_{n=2}^{\infty} S_n t^{-n} \tag{12}$$

which may be calculated from

$$D(g(t)) = D(t)/\{g'(t)\}. \tag{13}$$

LEMMA 5. (Baker [2]). *If the series* (3) *is embeddable and if the* $f_n(z)$ *are single-valued in their whole domain of existence,* $n = 1, 2, 3, \ldots$, *then there exists* $\rho_0 < 0$ *such that, for* z *in any annulus of the form* $0 < \rho_2 \leqslant |z| \leqslant \rho_1 < \rho_0$ *one has for all large enough* n

(i) $f_n(z)$ *regular and*

(ii) $f_n(z) \to 0$ *uniformly as* $n \to \infty$.

3. *Proof of Theorem* 1

We first prove

LEMMA 6. *Suppose* D *and* f *satisfy the assumptions of Theorem* 1. *Then* (a) *if* $f_n(z)$ *is analytically continuable (with at most algebraic singularities) from its expansion*

$$f_n(z) = z+n\, a_2 z^2+\ldots \tag{14}$$

at 0, *along a path* γ *of* D, *so is* $f_j(z)$ *for all* $j < n$, *and* $f_j(\gamma) \subset D, j < n$; (b) $f_n(z), n \geqslant 1$, *is single-valued as far as continuable analytically from* (14) *by paths lying completely in* D.

Proof. The proof that follows is by induction.

Case $n = 2$

Given $f(z) = z+a_2 z^2+\ldots$, with (i), (ii) and (iii) of Theorem 1 satisfied we consider the analytic continuation of $f_2(z)$ from $f_2(z) = z+2a_2 z^2+\ldots$, at 0. If we can analytically continue $f_2(z)$ along a path γ in D, starting at 0, then $f(\gamma) \subset D$; for otherwise there exists $p \in D \cap \gamma$ such that $f(p) \in \delta$, where δ is the boundary of D. We suppose p to be the first such point on γ starting from 0. Then

$$f(z) = f_2(f_{-1}(z)), \tag{15}$$

where $f_2(z) = z+2a_2 z^2+\ldots$, and $f_{-1}(z) = -za_2 z^2+\ldots$ near 0. If we consider (15) as z runs along $f(\gamma)$ from 0 to $f(p)$, a suitable branch of $f_{-1}(z)$ traverses γ from 0 to p and at $f(p)$ there is a branch of $f_{-1}(z)$ having at most an algebraic singularity such that $f_{-1}(f(p)) = p$. Thus in fact (15) gives a continuation of $f(z)$ having at most an algebraic singularity over its natural boundary at $f(p)$. This is impossible and hence $f(\gamma) \subset D$ for any path γ in D on which $f_2(z)$ is regular.

Suppose now $f_2(z)$ can be analytically continued from 0 to z_1 by two paths γ_1, γ_2 in D. Then $\gamma_1 \circ \gamma_2^{-1}$ is a path from 0 to 0 lying completely in D. As z traverses this closed path $f(z)$ traverses another closed path $\bar{\gamma} = f(\gamma_1) \circ f(\gamma_2)^{-1}$ which lies completely in D (by the preceding argument). As w traces $\tilde{\gamma}$, $f(w)$ is regular and continues from the branch $f(w) = w+a_2 w^2+ \dots$ at 0 to the same branch at the end of the path. Thus $f_2(z) = f(f(z))$ continues from the initial branch

$$f_2(z) = z+2a_2 z^2+\dots$$

at $z = 0$ back to the same branch at the end of the path as z traces $\gamma_1 \circ \gamma_2^{-1}$. Hence continuation of $f_2(z)$ from 0 along γ_1 or γ_2 to z_1 yields identical results.

General inductive step. Assume the statements of the lemma have been established for $2 \leqslant n < m$. We shall prove that they then hold for $n = m$.

We suppose that $f_m(z)$ is continuable from 0 along γ in D and that q is the first point of γ (starting from 0) at which the analytic continuation of f_{m-1} breaks down. The part of γ between 0 and q is denoted by γ'. By the induction hypothesis, f_j is regular on γ' and $f_j(\gamma') \subset D$, $j < m-1$, and further, since f_m, f_{m-1} are regular on γ' it is easy to see from $f_m = f(f_{m-1})$ that $f_{m-1}(\gamma') \subset D$. Let t, $2 \leqslant t \leqslant m-1$, be the smallest positive integer such that the continuation of f_t from 0 along γ breaks down at q. Thus $f_{t-1}(\gamma') \subset D$ and since $f_{t-1}(q) \in D$ would imply that f_t can be continued along γ over q, it follows that $f_{t-1}(q) \in \delta$.

We show next that $f_{m-1}(z)$ tends uniformly to δ as $z \to q$ on γ'. For if such is not the case there is a sequence $z_j \in \gamma'$, $j = 1, 2, \dots$, such that $z_j \to q$ while $w_j = f_{m-1}(z_j)$ tends to a limit $r \in D$. Since $f(w)$ is regular at $w = r$ we have

$$f_m(z_j) = f(f_{m-1}(z_j)) = f(w_j) \to f(r) = f_m(q) = p,$$

say. If we denote by $f_{-1}(w)$ the branch(es) of the inverse of $f(z)$ obtained by inverting the Taylor series $w-p = f(z)-p = f'(r)(z-r)+ \dots$, so that $f_{-1}(p) = r$, then $f_{-1}(w)$ is regular in a neighbourhood of p except for at worst a branch point at p, and $F(z) = f_{-1}(f_m(z))$ is regular in a neighbourhood of q (except for at worst a branch point) and one of its branches will agree near z_j with f_{m-1}. Thus $F(z)$ yields an (algebraic) continuation of f_{m-1} along γ' over q and by the induction hypothesis this continuation is in fact single-valued. This contradicts the definition of q.

Now recollect that $f_{t-1}(\gamma') \subset D$, $f_{t-1}(q) =$ (say) $s \in \delta$ and that f_{t-1} is regular at q. It follows from the mapping properties of analytic functions that there is a disc M of centre q (which we may assume so small as to lie in the region of regularity of f_m and f_{t-1}), an arc σ of δ containing s, and an arc σ_1 in D with the following properties: σ_1 passes through q and is analytic except perhaps at q; f_{t-1} maps σ_1 bijectively on to σ; σ_1 divides M into two components one of which, say N, contains $\gamma' \cap M$ and is mapped homeomorphically by f_{t-1} to a neighbourhood of σ in D. For a point q'' of σ_1 near q (i.e. on the boundary of N) we may modify γ to γ'' inside N so that γ'' ends in q'' instead of q. Now f_m and f_{t-1} are regular on γ'' (including q'') while

$$f_{t-1}\{\gamma''-(q'')\} \subset D.$$

Thus f_t can be continued along $\gamma''-(q'')$. However, since $f_{t-1}(q'') \in f_{t-1}(\sigma_1) \subset \sigma \subset \delta$, it can be seen that f_t cannot be continued over q'' along γ''. It follows that f_{m-1} cannot be continued over q'' along γ'', for the induction hypothesis would then yield the continuability over q'' of f_t.

Thus we may replace q by q'', γ by γ'' in the above arguments and find in particular that $f_{t-1}(\gamma'') \subset D$. For any of the paths γ'', the path $\tilde{\gamma} = f_{t-1}\{\gamma''-(q'')\}$ is a path in

D along which f_{m-t+1} and therefore, by the induction hypothesis, f_{m-t} also are continuable. Since γ'' may be taken to be a curve approaching q'' in σ_1 from inside N in an arbitrary manner, $\tilde{\gamma}$ may be taken to be an arbitrary approach curve to σ from inside D. Since $f_{m-1}(z) = f_{m-t}(f_{t-1}(2))$ approaches δ as $z \to q''$ in γ'' we see that $f_{m-t}(w)$ approaches δ as $w \to \sigma$ from inside D. The boundary values of f_{m-t} on the arc σ must be on one of the connected components of δ, i.e. on an analytic curve. By the Schwarz reflection principle f_{m-t} is therefore continuable across $\sigma \ni f_{t-1}(q)$ and so $f_{m-1}(z) = f_{m-t}\{f_{t-1}(z)\}$ is continuable along the arc γ' over q. This contradicts the assumptions and so we have proved that all f_j, $j < m$, can be continued along γ.

We have also to show that $f_j(\gamma) \subset D$ for $j < m$. It has just been shown that f_{m-1} can be continued along γ; so by the induction hypothesis $f_j(\gamma) \subset D$ for $j < m-1$. In particular, $\beta = f_{m-2}(\gamma) \subset D$. Now β is a path in D which starts at 0 and f_2 can be continued on β since f_m can be continued on γ. Thus $f_{m-1}(\gamma) = f(\beta) \subset D$, by the case $n = 2$.

Suppose finally that $f_m(z)$ can be continued analytically from $f_m(z) = z + ma_2 z^2 + \ldots$ at 0 by two paths γ_1 and γ_2 each leading to $z_1 \in D$. Then $\gamma_1 \circ \gamma_2^{-1}$ is a path from 0 to 0 lying completely in D. Now f_{m-1} can be continued on γ_1 and γ_2 to z_1 and is single-valued; so f_{m-1} can be continued round $\gamma_1 \circ \gamma_2^{-1}$ and leads back to its initial branch at 0; further, $f_{m-1}(\gamma_1 \circ \gamma_2^{-1}) \subset D$. Hence $f_m(z) = f\{f_{m-1}(z)\}$ can be continued round $\gamma_1 \circ \gamma_2^{-1}$ and leads back to its initial branch at 0, since f is single-valued; i.e. continuation of f_m along γ_1 or γ_2 leads to the same result at z_1.

The induction is now complete and the lemma established.

Proof of Theorem 1. We consider D and $f(z)$ which satisfy the assumptions of Theorem 1 and suppose $f(z)$ to be embeddable. We move our fix point to infinity; applying the usual transformation $z = k/t$, $z_1 = k/t_1$, we get corresponding to $f(z)$

$$g(t) = t+1+\sum_{k=1}^{\infty} b_k t^{-k}, \tag{15}$$

while to $f_n(z)$ corresponds

$$g_n(t) = t+n+\sum_{k=1}^{\infty} b_k(n)\, t^{-k} \tag{16}$$

with the usual properties. Then $g(t)$ is embeddable; also by Lemma 6 all $g_n(t)$ are single-valued as far as analytically continuable from an expansion about infinity within the image of D under $z \to t = kz^{-1}$ and so within a certain fixed neighbourhood $|t| \geqslant T$ of ∞ (independent of n).

We now consider $\mathscr{D}(K)$ as defined in Lemma 1 and choose $K(\geqslant T)$ so large that Lemma 2 holds and such that $A'(t)$ is regular in $|t| > K$ (*cf.* Lemma 4). By choosing K large enough we may suppose $A'(t)$ to be uniformly close to 1 in $\mathscr{D}(K)$. Then $w = A(t)$ maps $\mathscr{D}(k)$ univalently and conformally to a region C of the w plane lying to the right of a curve which approaches ∞ in directions $\arg w = \pm 3\pi/4$. C contains a half-plane $\operatorname{Re} w < B$. We now take $R_0 > K$, $R_0 < R_1 < R_2$. Let $R_1 < r < R_2$ and γ be the segment $t > r$ of the real axis, β the semi-circle $t = re^{i\theta}$, $0 \leqslant \theta \leqslant \pi$. Now $A'(t)$ is regular on $\beta \cup \gamma$ and $A(t)$ may be continued regularly along $\beta \cup \gamma$ to $t = -r$ with the values of $A(\beta)$ being bounded. For large enough n, $A(\beta)+n$ lies in $\operatorname{Re} w > B$, while, for all positive n, $t \in \gamma \subset \mathscr{D}(k)$ implies $A(t)+n = A(g_n(t)) \in F$. Thus $A(\beta \cup \gamma)+n$ is a curve in C.

Consider now $h(t) = A_{-1}\{A(t)+n\}$ on $\beta \cup \gamma$. On γ, $h(t) = g_n(t)$ while as t describes $\beta \cup \gamma$, $A(t)+n$ describes $A(\beta \cup \gamma)+n$ in C and the inverse of the univalent map $A : \mathscr{D}(k) \to C$ gives a regular continuation $h(t)$ of $g_n(t)$ along β to $-r$. Moreover, for $t = re^{i\theta}$, $0 \leqslant \theta \leqslant \pi$, and $R_1 \leqslant r \leqslant R_2$, $g_n(t)$ lies in a compact subset of $\mathscr{D}(k)$. Similarly considering a path $\beta'+\gamma$, $\beta' = re^{i\theta}$, $0 \geqslant \theta \geqslant -\pi$, we conclude that we can get a regular continuation of $g_n(t)$ along β' to $-r$. Now since $g_n(t)$ is single-valued as far as continuable within $|t| \geqslant T$ and we have chosen $R_1 > T$, the upper and lower continuations yield identical results, for $g_n(t)$.

Thus for large enough n $g_n(t)$ is regular in the annulus $R_1 \leqslant t \leqslant R_2$ and it maps the annulus to a compact subset of $\mathscr{D}(K)$ and further by Lemma 1, $g_n(t) \to \infty$ as n tends to infinity, locally uniformly in $\mathscr{D}(K)$ and hence uniformly in $R_1 \leqslant t \leqslant R_2$. Now by Lemma 6, if $g_n(t)$ is regular for all $n \geqslant N_0$, say, in *any* annulus in the neighbourhood of infinity, then so is it for all $n < N_0$. Hence for all n, there exists R_1 so that, for $|t| > R_1$,

$$g_n(t) \text{ is regular for all } n, \tag{17}$$

$$g_n(t) \to \infty \text{ uniformly as } n \to \infty. \tag{18}$$

Transferring our fix point to the origin once more, we see that there exists a $\rho_0 > 0$, such that in some $|z| < \rho_0$

$$f_n(z) \text{ is regular for all } n, \text{ and} \tag{19}$$

$$f_n(z) \to 0 \text{ uniformly as } n \to \infty. \tag{20}$$

Now, considering our expansion $f(z) = z + a_2 z^2 + \ldots$ near 0, we claim that $\{f_n(z)\}$ cannot form a normal family in any $|z| < \rho_0$. If it could, there would exist a subsequence $\{f_{n_k}(z)\}$ uniformly convergent in $|z| \leqslant \rho < \rho_0$; that is, $f_{n_k}(z) \to f(z) \equiv 0$ by (20).

This implies that $f'_{n_k}(0) \to 0$ contradicting the fact that $f'_{n_k}(0) = 1$. Hence $f(z)$ cannot be embeddable.

4. *Proof of Theorems* 2 *and* 3

The proof of Theorem 2 follows that of Theorem 1 except that Lemma 6 is replaced by

LEMMA 7. *If D and f satisfy the assumptions of Theorem* 2 *then the assertions* (a) *and* (b) *of Lemma* 6 *hold.*

The proof is contained in [6] and will not be given here.

Proof of Theorem 3. The case when f is meromorphic in the plane has been dealt with in [2] and covers the rational case in particular. We may therefore assume that f has an essential singularity at, say, a. If $T(z) = -az/(z-a)$, then

$$g = T \circ f \circ T^{-1}(z) = z + a_2 z^2 + \ldots,$$

has an essential singularity at ∞. Moreover, f is embeddable in a family

$$f_\lambda = z + \lambda a_2 z^2 + \ldots,$$

if and only if g is embeddable in a family $g_\lambda = T \circ f_\lambda \circ T^{-1} = z + \lambda a_2 z^2 + \ldots$. Clearly g is single-valued and has essential singularities at points $T(z')$ where z' are the essential singularities of f.

The iterates g_n are single-valued and regular except at the countable set of points formed by

$$\bigcup_{m=0}^{n-1} g_{-m}(s),$$

where s belongs to the set of singularities (including poles) of g. Once g_k has an essential singularity or pole at z', then so does g_n for $n > k$.

Suppose g is embeddable; then Lemma 5 shows that there exists ρ_0 such that for any $0 < \rho_1 \leqslant |z| \leqslant \rho_2 < \rho_0$ and all large enough n, g_n is regular in $\rho_1 \leqslant |z| \leqslant \rho_2$. Hence all g_n are regular in $0 < |z| < \rho_0$. The lemma also states that $g_n(z) \to 0$ uniformly in $\rho_1 \leqslant |z| \leqslant \rho_2$ and hence on $|z| \leqslant \rho_2$, by the maximum modulus theorem. But this implies $g_n'(0) \to 0$ which contradicts $g'_n(0) = 1$. Hence the theorem is proved.

5. *Examples*

Construction of Example 1. Let δ be the circumference $|z| = 1$, Δ the domain $|z| < 1$. Let $w = A(z) = \lambda/z + a_1 z + a_2 z^2 + \ldots$ map Δ univalently onto the exterior of an everywhere non-analytic Jordan curve Γ. (A is the Abel function of our group; *cf.* [1].) If a is large enough, $a > d$ say, where $d =$ diameter of Γ, then $\Gamma, \Gamma - a, \Gamma + a$ are disjoint. So $\Gamma \pm a$ lie in the exterior of Γ. Let the image of $\Gamma - a$ in Δ under $w = A(z)$ be γ [γ will be a non-analytic curve with 0 outside it] and D be the region between γ and δ. Consider now

$$f(z) = A_{-1}\{A(z) + a\} = z + b_2 z^2 + \ldots, \text{ near } z = 0.$$

In fact

$$A_{-1}(w) = z = \frac{\lambda}{w} + \frac{a_1 \lambda^2}{w^3} + \ldots,$$

and

$$f(z) = A_{-1}(A(z) + a) = z - \frac{a}{\lambda} z^2 + \frac{a^2}{\lambda^2} z^3 + \ldots .$$

(By construction, a is non-zero and so $b_2 = -a/\lambda \neq 0$.)

Now $f(z)$ is defined in D and univalent there. We can continue $f(z)$ along any curve in D surrounding γ. Moreover, as z tends to the analytic curve δ from within D, $A(z) + a \to \Gamma + a$ and $f(z) \to$ non-analytic $A_{-1}(\Gamma + a)$. Also as z tends to the non-analytic curve γ in Δ, $A(z) + a \to \Gamma$ and $f(z) \to$ analytic δ. Hence f is not continuable over δ or γ. Further, f is embeddable in the family of iterates

$$f_\mu(z) = A_{-1}(A(z) + \mu a) = z - \frac{\mu a}{\lambda} z^2 + \ldots,$$

which has $f_1(z) = f(z)$ and a positive radius of convergence for each μ. Such a family satisfies $f_\mu \circ f = f \circ f_\mu = f_{\mu+1}$ and is thus the unique family associated with f. We note that the assumptions of Theorem 1 are violated in that the boundary of D is not wholly analytic.

As an illustration of Theorem 2 we discuss

Example 2. Let δ be any Jordan curve whose interior, D, is a region containing the origin and let λ, μ be the minimum and maximum distances of δ from the origin, respectively. Now consider $f(z) = a_0 + a_1 z + \ldots$, analytic in D such that $a_0 \neq 0$ and $f(z)$ is not continuable across δ. We suppose further that $w = f(z)$ has boundary values all in $|w| > \theta > 0$, say, on δ. Then $g(z) = z + kz^2 f(z)$ is also analytic in D, with natural boundary ε. For suitable choice of k, the boundary values of g on δ satisfy

$$|g(z)| > k\lambda^2 \theta - \mu > 2\mu$$

and are therefore outside D. Also $g = z + ka_0 z^2 + ka_1 z^3 + \ldots$, $ka_0{}^2 \neq 0$; so, by Theorem 2, $g(z)$ is non-embeddable.

6. *Proof of Theorem* 4

Given $f(z)$ of the form (3), convergent in some neighbourhood of 0, we assume that the set of λ for which the corresponding fractional iterates f_λ in (4) have positive radius of convergence contains a two-dimensional lattice L. We select a fixed $\lambda \neq 0$ in L such that $\frac{2}{3}\pi < \theta = \arg \lambda < \frac{3}{4}\pi$.

Making the transformation $z = kt^{-1}$, $z_1 = kt^{-1}$, $z_\lambda = kt_\lambda{}^{-1}$, where $a_2 k = -1$, shifts the fixed point to ∞ and replaces $z_1 = f(z)$ by

$$t_1 = g(t) = t + 1 + \sum_1^\infty b_n t^{-n} \tag{5}$$

and $z_\lambda = f_\lambda(z)$ by

$$t_\lambda = g_\lambda(t) = t + \lambda + \sum_1^\infty b_n(\lambda) t^{-n}, \tag{7}$$

convergent in a neighbourhood of ∞.

We choose K so large that the assertions of Lemmas 1 and 2 hold in the set $\mathscr{D}(K)$ defined in Lemma 1. Since $\frac{2}{3}\pi < \theta = \arg \lambda < \frac{3}{4}\pi$ while the boundaries of $D(K)$ run to ∞ in the directions $\arg t = \pm \frac{3}{4}\pi$ and since by (7) $g_\lambda \approx t + \lambda$ for large t, there exists a half-plane $H : \mathrm{Re}\{t \exp(-\frac{3}{4} i\pi)\} > M$ for a suitably large M, such that $H \subset \mathscr{D}(K)$ and $g_\lambda(H) \subset H$. It follows that for $t \in H$ and the function $A(t)$ of Lemma 2,

$$\begin{aligned} A\{g_\lambda(t)\} &= \lim_{n \to \infty} \{g_n(g_\lambda(t)) - n - b_1 \log n\} \\ &= \lim_{n \to \infty} \{g_\lambda(g_n(t)) - n - b_1 \log n\} \\ &= A(t) + \lambda_1 \end{aligned} \tag{21}$$

by (10) and (7), and the fact that $g_n(g_\lambda) = g_\lambda(g_n)$.

Differentiating (21) and using $g_\lambda(H) \subset H$, we have in H

$$\begin{aligned} A'(t) &= A'(g_\lambda(t))\, g_\lambda(t) \\ &= A'\{g_{N\lambda}(t)\} \prod_{n=0}^{N-1} g_\lambda{}'\{g_n(t)\}. \end{aligned} \tag{22}$$

Now in (7) $\lambda = |\lambda| e^{i\theta}$ and the transformations $\tau_1 = t_\lambda \lambda^{-1}$, $\tau = t\lambda^{-1}$ change (7) into

$$\tau_1 = \tau + 1 + b_1 \lambda^{-1} \tau^{-1} + \sum_2^\infty b_n{}' \tau^{-n} = h(\tau). \tag{23}$$

The results of Lemmas 1 and 2 may then be applied to $h(\tau)$, so that there is a K' such that in the domain $\mathscr{D}(K')$ we have $h_n(\tau) \subset \mathscr{D}(K')$, $h_n(\tau) \to \infty$ like $n + b_1 \lambda^{-1} \log n$

and so on. This means that for t in the domain $\mathscr{D}' = \lambda\mathscr{D}(K')$, $g_{n\lambda}(t) \in \mathscr{D}'$ and $g_{n\lambda}(t) \to \infty$. Moreover, there is a function $A_1(t)$ such that

$$A_1(t) = \lim_{n\to\infty} \{g_{n\lambda}(t) - n\lambda - b_1 \lambda^{-1} \log n\} \text{ in } \mathscr{D}', \tag{24}$$

while

$$A_1\{g_{n\lambda}(t)\} = A_1(t) + n\lambda \text{ in } D'. \tag{25}$$

Further, A_1 is regular and univalent in $\mathscr{D}'$ and $A_1'(t) \to 1$ as $t \to \infty$ in D'. Differentiating (25) and letting $n \to \infty$ we have therefore

$$A_1'(t) = \prod_{n=0}^{\infty} g_\lambda'\{g_{n\lambda}(t)\}, \qquad t \in \mathscr{D}'. \tag{26}$$

Noting that $g_\lambda(H \cap \mathscr{D}') \subset H \cap \mathscr{D}'$ we see that for $t \in H \cap \mathscr{D}' \subset \mathscr{D}'$, $g_{N\lambda}(t) \to \infty$ as $N \to \infty$ and the values $g_{N\lambda}(t) \in H \subset \mathscr{D}$ so that $A'\{g_{N\lambda}(t)\} \to 1$ and so (22) implies that

$$A'(t) = \prod_{n=0}^{\infty} g_\lambda'\{g_{n\lambda}(t)\}, \qquad t \in H \cap \mathscr{D}'. \tag{27}$$

Since $H \cap \mathscr{D}'$ is a non-empty sector, $A_1'(t)$ is an analytic continuation of $A'(t)$ by (26) and (27) into a region of the form $\mathscr{D}' = \lambda\mathscr{D}(K')$.

By repeating the above arguments with the pair (g_1, g_λ) replaced successively by (g_λ, g_{-1}), $(g_{-1}, g_{-\lambda})$, $(g_{-\lambda}, g_1)$ we see that the functions $A'(t)$, $A_1'(t)$

$$A_2'(t) = \prod_0^{\infty} g'_{-1}\{g_{-n}(t)\}$$

$$A_3'(t) = \prod_0^{\infty} g'_{-\lambda}\{g_{-n\lambda}(t)\}$$

and $A'(t)$ are regular in domains $\mathscr{D}(K)$, $\lambda\mathscr{D}(K')$, $-\mathscr{D}(K'')$, $-\lambda\mathscr{D}(K''')$, $\mathscr{D}(K)$ respectively, each neighbouring pair of which overlap, and the corresponding functions are identical in the regions of overlapping. Thus $A'(t)$ may be contained analytically in a punctured neighbourhood of ∞ and is single-valued in this neighbourhood. Since $A'(t) \to 1$ as $t \to \infty$ the point at ∞ is a removable singularity of $A'(t)$.

Thus there exists $A'(t)$ regular at ∞ and satisfying (*cf.* (11))

$$A'\{g(t)\}\, g'(t) = A'(t) \tag{28}$$

where

$$g(t) = t + 1 + b_1 t^{-1} + \dots \tag{29}$$

Since $A_1'(\infty) = 1$, calculation of (28), (29) gives

$$A'(t) = 1 - b_1 t^{-1} + ct^{-2} + \sum_{n=3}^{\infty} c_n t^{-n}. \tag{30}$$

Now in fact the converse of Lemma 4 holds, i.e. we can conclude from the existence of $A'(t)$ that g (and hence f) is embeddable. This has been proved independently by Erdős and Jabotinsky [5; p. 361–76] and by Baker [1; p. 289–290]. Indeed, set

$$A(t) = t - b_1 \log t - ct^{-1} - \sum_3^{\infty} \frac{c_n}{(1-n)} t^{1-n} \tag{31}$$

With $w = t(1+v)$ and for *arbitrary* constant λ, put

$$A(w) = A(t)+\lambda; \tag{32}$$

this gives

$$v-b_1 t^{-1} \log(1+v) - ct^{-2}(1+v)^{-1} + \ldots = \lambda t^{-1} - ct^{-2} + \ldots, \tag{33}$$

which is satisfied for $t^{-1} = 0$, $v = 0$. Taking $\log(1+v) = v - \frac{1}{2}v^2 + \ldots$, both sides of (33) are analytic near $t^{-1} = 0$, $v = 0$ and the derivative of the left-hand side for v is 1 at $t^{-1} = 0$, $v = 0$. Thus there is a solution $v = \lambda t^{-1} + \ldots$ regular at $t = \infty$, which corresponds to the solution

$$w = h_\lambda(t) = t + \lambda + \sum_1^\infty d_n t^{-n} \tag{34}$$

of (32). The functions $h_\lambda(t)$ exist for each λ, and satisfy (32) for large t and a suitable determination of the logarithm. Hence, by differentiation,

$$A'\{h_\lambda(t)\} h_\lambda'(t) = A'(t). \tag{35}$$

But comparison of coefficients shows that to given A' there is a unique series (34) beginning $w = t + \lambda + \ldots$ which satisfies (35). It follows from (28) that $h_1(t) = g(t)$, and to given λ, μ that (for large t)

$$A'[h_\lambda\{h_\mu(t)\}] h_\lambda'\{h_\mu(t)\} h_\mu'(t) = A'\{h_\mu(t)\} h_\mu'(t) = A'(t),$$

so that

$$h_\lambda\{h_\mu(t)\} = h_{\lambda+\mu}(t) = h_\mu\{h_\lambda(t)\}.$$

Thus $h(t)$ are the commuting family of iterates $h_\lambda(t) \equiv g_\lambda(t)$ associated with $g(t) = g_1(t)$ so $g(t)$ (and consequently also $f(z)$) is embeddable.

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