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A SURVEY OF THE THEORY OF FUNCTIONAL EQUATIONS*

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Introduction

When a few years ago I wrote in an expository article on functional equations (Kuczma [11]) „We must be aware, however, that in view of the very quick development of this theory the situation here described can become inactual within a few months" I did not think that these words would come true so soon. Now, starting to write the present paper, I have realized that I can make use of that article but a little and a great deal must be written anew. Similarly, the English edition of J. Aczél's book (now already under the press) will contain almost twice as much material as the first one (Aczél [19]).

But the theory of functional equations is relatively young. The first paper (if we do not take into account a use of recurrences, which reaches as far back as Archimedes; cf. Pincherle [1]) dealing with functional equations[1] (d'Alembert [1]) was written in 1747. Since then a number of mathematicians (among them as eminent as e.g. Abel, Cauchy, Gauss, Euler) would write single papers devoted to this or that particular functional equation. Probably A. R. Schweitzer was the first who has made an attempt to treat the subject more uniformly. He also planned to gather a bibliography of functional equations (Schweitzer [2]). But the beginning of *a theory of functional equations* is connected with the work of an excellent specialist in this field, Hungarian mathematician J. Aczél. In his numerous papers he treats whole classes of functional equations, gives general methods of solving functional equations and criteria of the existence and uniqueness of solutions. He also indicates many new applications of functional equations.

This young theory is now rapidly developing. The number of mathematical papers dealing with functional equations is still increasing. In last years three monographs on functional equations have been written (Aczél [19], Aczél-Gołąb [1], Ghermănescu [18]; cf. also the booklet Aczél [24]), which have gained a great popularity. J. Aczél's book has recently been translated into English, and a sudden death in 1962 has interrupted M. Ghermănescu the work on a French translation of his book. A book by J. Anastassiadis concerned with defining Euler's functions by functional equations (Anastassiadis [7]) will appear soon. Also the author of the present paper is preparing a monograph on functional equations in a single variable. Moreover every year we observe some new books on finite differences and difference equations.

Another sign of the growing importance of functional equations is the fact that a number of mathematicians have devoted themselves mainly or entirely to the research work in this branch of the mathematics. Including also the mathematicians who have been led to functional equations by their investigations in other subjects (like differential geometry, iterations and analytic functions, differential equations, number theory, abstract algebra) and who have not once contributed to functional equations, we would like to mention here the names of J. Aczél, J. Anastassiadis, T. Angheluţă, M. Bajraktarević, I. N. Baker, L. Berg, S. Bochner, D. Brydak, K. Chandrasekharan, B. Choczewski, B. Crstici, Z. Daróczy, D. Ž. Đoković, I. M. H. Etherington, V. Ganapathy Iyer, O. E. Gheorghiu, S. Gołąb, W. Hahn, M. Hosszú, M. Kucharzewski, M. Kuczma, S. Kurepa, L. Losonczi, W. Maier, S. Mandelbrojt, M. A. McKiernan, D. S. Mitrinović, P. J. Myrberg, R. Narasimhan, S. Prešić, F. Radó, A. Sade, B. Schweizer, I. Stamate, G. Szekeres, M. Urabe, P. M. Vasić, E. Vincze and A. Zajtz. This list, of course, does not claim to be complete. For a few last years international conferences on functional equations have been held: in Balatonvilágos (1961), in Sárospatak (1963) and in Oberwolfach (1962 and 1963).

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* Presented by D. S. Mitrinović.

[1] It was the equation $\varphi(x+y)+\varphi(x-y)=\psi(x)\,\eta(y)$.

Of course one may ask what is the reason of this interest taken in functional equations by the mathematicians of all the world. This may be connected with the fact that in many branches of the mathematics analytical methods are already exhausted to some extent. A use of elementary methods (to which belong also functional equations) often allows one to obtain much deeper and more general results than it was possible with a use of classical methods of mathematical analysis. On the other hand, more and more problems of physics and technics requires making weak assumptions regarding the occurring functions. In such a case differential equations are often replaced by functional equations.

The present article is an expository one. We quote here the most important results of the theory of functional equations, however, omitting the proofs. For details the reader is referred to the books by J. Aczél [19], [24] and M. Ghermănescu [18], as well as to the research papers quoted in the end of this article. This bibliography contains a selection of papers on functional equations and is by no means complete. We have aimed at including most of the important papers of the recent period and a number of classical earlier items. We have not included research papers on geometric objects (which are often handled by methods of functional equations), as a bibliography concerning this subject may be found in the book by J. Aczél and S. Gołąb [1] and in the article by M. Kucharzewski and M. Kuczma [5]. Few exceptions are the papers where equations of a more general interest are treated. Here and there we mention some unsolved problems. Others are to be found in J. Aczél's book (Aczél [19]) and in two collections of problems published recently by J. Aczél [25] and D. S. Mitrinović-D. Ž. Đoković [10].

1. Definition of a functional equation

The first point one must agree upon when one starts to speak about a theory of functional equations is the exact meaning of the notion „a functional equation". Originally it had contained all the equations in which unknown functions occur and thus also differential, integral equations etc. But now the expression „a functional equation" is usually used in a more restricted sense. However, various authors give definitions of different comprehension (cf. e.g. Aczél-Kiesewetter [1], Ghermănescu [2]). The below definition is a slightly modified version of that from the monograph Aczél [19]. It is based on the concept of a term, so we start with defining the latter.

Definition 1. *A term* is defined by the following conditions:

1° Independent variables are terms.

2° If $t_1, \ldots, t_p$ are terms and $f(x_1, \ldots, x_p)$ is a p-place function (i.e. a function of p variables), then $f(t_1, \ldots, t_p)$ also is a term.

3° There exist no other terms.

Then a functional equation may be defined as follows:

Definition 2. *A functional equation* is an equality $t_1 = t_2$ between two terms t_1 and t_2 which contain at least one unknown function[2] and a finite number of independent variables. This equality is to be satisfied identically with respect to all the occurring variables in a certain set (of any sort).

The solution of a functional equation may depend quite essentially on the set in which the equation is postulated. E.g. if we require that equation (23) be satisfied for all $x, y \in [-1,1]$, then $\varphi(x) \equiv 0$ is the only solution (Gołąb-Losonczi [1], [2], Kiesewetter [3], cf. § 6).

One should also precisely state in what a function class the solution is sought. The number and behaviour of solutions depends very strongly on this class. It is one of the important differences between differential and functional equations. In the case of the formers the function class in which the

[2] In the present article we shall use Greek letters to denote unknown functions and Latin letters to denote variables and given functions.

solution is sought is determined by differentiability conditions regarding the unknown function.

The notion of a functional equation as defined above does not contain differential, integral, operator equations and generally equations in which infinitesimal operations are performed. Thus differential equations with a lag have also been excluded as well as equations occurring in the theory of dynamic programming (Bellman [1]), in which there appear maxima of some expressions. However, what is left is much enough to constitute a very large material in which a further division and specialization must be done.

2. Classification of functional equations

The problem of a classification of functional equations is very difficult and has not been solved till the present in a satisfactory manner. J. Aczél in his monograph follows the pattern: one or more unknown functions of one or more variables — altogether four types. Of course, this is a very rough classification; nevertheless it turns out useful.

Definition 3. A functional equation in which all the unknown functions are one-place functions (functions of one variable) is called *an ordinary functional equation*. A functional equation in which at least one of the uknown functions is a more-place function is called *a partial functional equation*.

Let us note that several functions can be completely determined by a single functional equation, contrary to the situation in differential equations.

A proposition of a classification of ordinary functional equations has been described in the paper Kuczma [25]. This classification is based on the notions of a rank[3], order and implication index.

The notion of the rank of a functional equation has been introduced by W. Maier [1].

Definition 4. The number of independent variables occurring in a functional equation is called *the rank* of this equation.

The above definition can also be applied to partial functional equations but in our opinion it is not appropriate as a base of a classification of partial equations.

A definition of the order of a functional equation has been given in papers Kuczma [11], [25]. For some special types of functional equations the order had previously been defined by M. Ghermănescu [2], [18]. Before we give here a precise definition we would like to call the reader's attention to some facts.

By suitable substitutions we may reduce a given functional equation to a system of equations in which under the sign of the unknown function only single variables (and not expressions built of variables) occur. So e.g. the Cauchy equation (Cauchy [1])

$$\varphi(x+y)=\varphi(x)+\varphi(y) \tag{1}$$

[3] We follow here the terminology of the paper Choczewski [1]. In the English translation of the book Aczél [19] rank is called order, and what here is named order does not occur at all. The original German words for rank, order and implication index are *Stufe, Ordnung* and *Schachtelungsexponent*, respectively.

may be written as

$$\varphi(z)=\varphi(x)+\varphi(y),$$
$$z=x+y;$$

the d'Alembert equation

$$\varphi(x+y)+\varphi(x-y)=2\varphi(x)\varphi(y) \tag{2}$$

may be written as

$$\varphi(u)+\varphi(v)=2\varphi(x)\varphi(y),$$
$$u=x+y, \quad v=x-y.$$

The equation (Gołąb-Schinzel [1])

$$\varphi[x+y\varphi(x)]=\varphi(x)\varphi(y) \tag{3}$$

may be written as

$$\varphi(z)=\varphi(x)\varphi(y),$$
$$z=x+y\varphi(x).$$

Here the additional equation $z=x+y\varphi(x)$ contains again the unknown function $\varphi(x)$. In the case of the equation (Gołąb [3], [4])

$$\varphi(xy)=x^n\varphi\left[y+\varphi^{-1}\left(\frac{m}{x^n}\right)\right] \tag{4}$$

the substitution must be made in two steps. The first leads to the system

$$\varphi(z)=x^n\varphi(w),$$
$$z=xy, \quad w=y+\varphi^{-1}\left(\frac{m}{x^n}\right),$$

where one of the additional equations contains the unknown function and moreover under the sign of the unknown function in this equation occurs not a single variable but the expression m/x^n. Therefore a new substitution is necessary, after which the system takes the form

$$\varphi(z)=x^n\varphi(w),$$
$$z=xy, \quad w=y+\varphi^{-1}(t),$$
$$t=\frac{m}{x^n}.$$

Here the equation $t=m/x^n$ cannot be written in the same line as the preceding two, since it is subordinate to those equations. It forms a second group of additional equations. Similarly in the case of the equation (Babbage [1])

$$\varphi(\varphi(\varphi(x)))=x \tag{5}$$

we have two groups of additional equations (each consisting of a single equation)

$$\varphi(z)=x,$$
$$z=\varphi(w),$$
$$w=\varphi(x).$$

Definition 4. The smallest number of additional equations which are necessary in order to reduce a functional equation to a from where under the sign of the unknown function only single variables occur, is called *the order* of this equation.

So e.g. the Cauchy equation has order 1, the d'Alembert equation has order 2. The order of equation (3) equals 1, order of equation (4) is 3 and that of equation (5) is 2. The most general ordinary functional equation with one unknown function, of order 1, has the form

$$F[x_1, \ldots, x_p, \varphi(x_1), \ldots, \varphi(x_p), \varphi\{f[x_1, \ldots, x_p, \varphi(x_1), \ldots, \varphi(x_p)]\}] = 0. \tag{6}$$

It may be reduced to the system

$$F[x_1, \ldots, x_p, \varphi(x_1), \ldots, \varphi(x_p), \varphi(y)] = 0,$$
$$f[x_1, \ldots, x_p, \varphi(x_1), \ldots, \varphi(x_p)] = y.$$

The above definition of the order has some shortcomings (cf. Kuczma [25]). It cannot be applied to partial functional equations. But even in the case of ordinary functional equations some ambiguities can arise. They may be caused by the requirement that the number of additional equations should possibly be the smallest. It is often difficult to decide whether it really is. E.g. the equation

$$\varphi(x+y) = \varphi(x) + y \tag{7}$$

has apparently order 1:

$$\varphi(z) = \varphi(x) + y,$$
$$z = x + y.$$

But in fact it is of order zero, since it may be written in the form

$$\varphi(z) = \varphi(x) + z - x,$$

where x and z are not connected by any relation. Similarly equation (6) has order 1 provided the function $F(x_1, \ldots, x_p, z_1, \ldots, z_p, u)$ really depends on each of the variables $z_1, \ldots, z_p, u$.

Roughly speaking, the implication index says how many times iterated is the unknown function in the equation.

Definition 5. Suppose that a functional equation has been reduced to a system of equations in the above described manner. The number of groups of additional equations which contain the unknown function is called *the implication index* of this equation.

One can unify the rank p, the order n and the implication index i of a functional equation into one symbol $[p, n, i]$ called the type of this equation. So equations (1), (2), (3), (4), (5), (6), (7) have types [2, 1, 0], [2, 2, 0], [2, 1, 1], [2, 3, 1], [1, 2, 2], $[p, 1, 1]$, [2, 0, 0], respectively.

Some theorems regarding the reduction of the rank have been proved by J. Aczél and H. Kiesewetter [1]. From their results it follows that rank 2 plays a particular rôle in the theory of functional equations in the sense that equations of a higher rank usually can be replaced by equivalent equations of rank 2 (e.g. the families of solutions of equation (1) and of the equation $\varphi(x_1 + \cdots + x_p) = \varphi(x_1) + \cdots + \varphi(x_p)$ are identical), while similar replacing an equation of rank 2 by an equation of rank 1 is in general not possible. The reduction of the order has been investigated by M. Kuczma [26].

The above described classification concernes only ordinary functional equations. To ordinary as well as partial equations one can apply the notion of *the grade* defined by A. R. Schweitzer [1] as $2j-p$, where j is the smallest number of variables on which depend the unknown functions (so $j=1$ for ordinary functional equations) and p is the rank. Still another approach to the classification problem has been proposed by B. Schweizer and A. Sklar [2].

All these attempts, however, do not prove satisfactory. Two functional equations with the same characteristics may differ by the structure of their solutions. E.g. the Cauchy equation (1) and the Jensen equation (Jensen [1])

$$\varphi\left(\frac{x+y}{2}\right)=\frac{\varphi(x)+\varphi(y)}{2} \tag{8}$$

both have the same type [2, 1, 0] and the same grade 0. Nevertheless equation (1) has a one-parameter family of continuous solutions

$$\varphi(x)=cx, \tag{9}$$

while equation (8) has a two-parameter family of continuous solutions

$$\varphi(x)=ax+b. \tag{10}$$

Similarly the equations

$$\varphi[x,\ \varphi(y,z)]=\varphi[\varphi(x,y),\ z] \quad \text{and} \quad \varphi[x,\ \varphi(y,z)]=\varphi[y,\ \varphi(x,z)]$$

have apparently the same form, while the solution of the former

$$\varphi(x,y)=f^{-1}[f(x)+f(y)]$$

contains one arbitrary function, $f(x)$, and the solution of the latter

$$\varphi(x,y)=f^{-1}[g(x)+f(y)]$$

contains two arbitrary functions, $f(x)$ and $g(x)$.

However, results of Z. Daróczy [1] (cf. § 5) show that there is not much chance of finding criteria which would allow us to decide from the outer look of a functional equation about the structure of its solutions. Here again we see a deep difference between functional and differential equations.

3. Methods of the theory of functional equations

The lack of general methods in the theory of functional equations had for long years been one of the causes that had discouraged mathematicians from this theory. The works of C. Popovici [1] and M. Ghermănescu [1], [2], [5] changed the situation for the better. But the papers by J. Aczél [9], [15] were a real progress. J. Aczél gave general methods of solving wide classes of functional equations, as e.g.

$$\varphi(x+y)=F[\varphi(x),\ \varphi(y)], \tag{11}$$

$$\varphi\left(\frac{x+y}{2}\right)=F[\varphi(x),\ \varphi(y)], \tag{12}$$

$$\varphi(ax+by+c)=F[\varphi(x),\ \varphi(y)], \tag{13}$$

$$G[\varphi(x+y),\ \varphi(x-y),\ \varphi(x),\ \varphi(y),\ x,\ y]=0 \tag{14}$$

etc. He gave also criteria of the existence and uniqueness of solutions (cf. § 5). Since then further general methods have been found by J. Aczél and his disciples; we must mention here the nice determinant method of E. Vincze [6]. For equations of rank 1 a number of general results have been established by M. Ghermănescu (cf. Ghermănescu [18]) and by the representatives of the Kraków school: B. Choczewski, J. Kordylewski and M. Kuczma (cf. §§ 14—22).

It would be impossible to describe all the methods used at solving particular functional equations. Very generally speaking, one may say that in the case of equations of rank $\geqslant 2$ the most frequently employed method is that of a specialization of variables. So e.g. setting $x=0$ in (7) yields immediately the solution $\varphi(x)=x+c$. In most cases, however, the solution cannot be obtained in such a simple way and the process of a specialization must be repeated several times in a rather ingenious manner.

The method of a specialization of variables cannot be used in the case of equations of rank 1. This part of the theory requires a completely different approach. To often employed methods belong: an extension of a function defined on a certain set to a solution of the equation in question, deriving the form of the solution (usually fulfilling some additional conditions) from the form of the equation, applications of fixed-point theorems in function spaces. So e.g. it is evident that any function defined on [1, 2) can be uniquely extended to a solution of the equation

$$\varphi(x+1)-\varphi(x)=\frac{1}{x^2}, \qquad x\in(0, \infty). \tag{15}$$

Further, from (15) the formula

$$\varphi(x+n)-\varphi(x)=\sum_{k=0}^{n-1}\frac{1}{(x+k)^2}$$

can be derived, whence it follows that the function $\varphi(x)=-\sum_{k=0}^{\infty}\frac{1}{(x+k)^2}$ is the unique solution of equation (15) fulfilling the condition $\lim_{x\to\infty}\varphi(x)=0$. Lastly, the only continuous solution $\varphi(x)=x$ of the equation

$$\varphi\left(\frac{x}{2}\right)=2\varphi(x)-\frac{3}{2}x, \qquad x\in[-1, +1],$$

can be obtained as the unique fixed point of the contraction map

$$T(\varphi)=\frac{1}{2}\varphi\left(\frac{x}{2}\right)+\frac{3}{4}x$$

of the space of continuous functions on $[-1, +1]$ into itself. Of course, in most cases the argument is more involved.

But a certain general method has been known and used for years. It consists in reducing functional equations to differential equations (cf. e.g. Aczél [3]). Its principles had been explained already by N. H. Abel [2], whose reasoning has recently been given a new, precise form by H. Kiesewetter [1]. This method, although very general, has a serious defect: it yields only differentiable (often even several times differentiable) solutions of the equation considered. I. Fenyö [1] tries to overcome this difficulty. The main idea of his interesting paper can be described as follows.

The original functional equation is considered as an equation for distributions. As is well known, distributions always have derivatives of all orders.

Thus the original equation can be reduced without difficulty to a differential equation for distributions. This is solved and afterwards it can be proved that the resulting distributions are functions.

Thist last step requires, however, some integrability assumptions. Now, there are known theorems regarding the differentiability of integrable solutions of certain functional equations (e.g. Kac [1], Aczél [22], [19]). Nevertheless the method of I. Fenyö is more general and, moreover, it may be regarded as a general method of solving distributional equations.

Finally, let us also mention that I. Carstoiu [1] has remarked that integral transforms can be applied to solving functional equations and A. Rényi [1] gave a method of reducing functional equations to integral equations.

Nonetheless the situation in the theory of functional equations is still far from that we observe in the theory of differential equations. But also the variety of problems connected with functional equations is much greater.

4. The Cauchy equations

Undoubtly the most widely known functional equation is the Cauchy equation[4]

$$\varphi(x+y)=\varphi(x)+\varphi(y). \tag{1}$$

This equation finds applications almost in every branch of mathematics. It plays an important part in the mechanics (Darboux [1], Schimmack [1]) and in the projective geometry (Darboux [2]). A. Cauchy [1] proved that the general continuous solution of equation (1) is given by formula (9). In the above theorem the condition of the continuity of φ can be considerably weakened (Sierpiński [2], Kac [1], Alexiewicz-Orlicz [1], Ghermănescu [10], Kuczma [17]). The most general result in this direction (Ostrowski [1], Kestelman [1]) is to the effect that (9) *is the only solution of equation* (1) *which is bounded from one side on a set of a positive measure.* The existence of discontinuous solutions of equation (1) was proved (with a use of the axiom of choice) by G. Hamel [1].

The case where equation (1) is satisfied not for all x, y has been investigated by J. Aczél [17] and S. Hartman [1]. The latter considered (1) in connection with the following problem of P. Erdös. Suppose that a function $\varphi(x)$ satisfies (1) for almost all pairs (x, y) of real numbers, is it true that $\varphi(x)$ is then equal almost everywhere to a function which satisfies (1) for all x, y? This problem remains still unsolved, but S. Hartman proved that if $\varphi(x)$ satisfies (1) for every x, y belonging to a linear set whose complement has measure zero, then $\varphi(x)$ satisfies (1) for all x and y.

The Cauchy equation is also connected with the so called difference property. A function class Ω is said to have *the difference property* (after de Bruijn [1]), when every function f such that $f(x+h)-f(x)\in\Omega$ for each h can be represented as a sum $f=g+\varphi$, where $g\in\Omega$ and φ satisfies (1). For numerous classes of functions the difference property has been proved by N. G. de Bruijn [1], [2], J. H. B. Kemperman [1] and F. W. Carroll [1].

[4] Cauchy [1]. Before Cauchy equation (1) was treated by A. M. Legendre (1791) and C. F. Gauss (1809).

Concerning the Cauchy equation we would like to mention also the following problem[5]. Is every solution $\varphi(x)$ of equation (1) such that $\varphi\left(\frac{1}{x}\right)=\frac{1}{x^2}\varphi(x)$ for all $x\neq 0$ necessarily of form (9)?

Related equations

$$\varphi(x+y)=\varphi(x)\,\varphi(y), \tag{16}$$

$$\varphi(xy)=\varphi(x)+\varphi(y), \tag{17}$$

$$\varphi(xy)=\varphi(x)\,\varphi(y), \tag{18}$$

are also called Cauchy equations. They can be easily reduced to equation (1). Their general measurable solutions are[6]

$$\varphi(x)=e^{cx},\quad \varphi(x)\equiv 0, \tag{16'}$$

$$\varphi(x)=c\log|x|,\quad \varphi(x)\equiv 0, \tag{17'}$$

$$\varphi(x)=|x|^c,\quad \varphi(x)=|x|^c\operatorname{sgn}x,\quad \varphi(x)\equiv 1,\ \varphi(x)\equiv 0, \tag{18'}$$

respectively. In the real domain $\varphi(x)=x$ is the only nontrivial ($\varphi(x)\not\equiv 0$) function that satisfies simultaneously equations (1) and (18) (Mineur [1]). In the complex domain $\varphi(x)=\bar{x}$ (x conjugate) is the other nontrivial solution of (1) and (18) (Noether [1]).

The Cauchy equations find applications in the mathematics of finances (Aczél [5]), in the probability theory (Gauss and Poisson distributions; cf. Császár [1]) and in many other topics. In the case where the arguments and/or the values of the function φ lie in abstract sets equation (18) plays an important part in algebra as the equation of isomorphisms, homomorphisms etc. (cf. also § 11; some generalizations are to be found also in Aczél [18], [21]).

The Jensen equation (8) (Jensen [1]) has many properties analogous to those of equation (1). Its general solution bounded from one side on a set of a positive measure is given by (10). J. Aczél and I. Fenyö [1] have applied equation (8) to define the centre of gravity of fields of forces. Further applications of equation (8) are to be found in the papers Aczél [5], Bajraktarević [2], [6].

5. Generalizations

Equation (11) may be regarded as a natural generalization of equation (1). It has been dealt with by several authors (Montel [1], Alt [1], Dunford-Hille [1], Thielman [1], Kuwagaki [1], Aczél [9], [13], [17] and others). J. Aczél has proved that *equation* (11) *has a continuous and strictly monotonic solution if and only if the function* $F(u, v)$ *is continuous and strictly monotonic with respect to each variable and fulfils the condition*

$$F[F(u, v), w]=F[u, F(v, w)].$$

(In other words, *a necessary and sufficient condition that equation* (11) *possess in* $(-\infty, +\infty)$ *a non-constant continuous solution taking values from an interval*

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[5] Oral communication by Professor I. Halperin. (Note added in proofs: We have been informed that this problem has recently been solved in the positive by W. B. Jurkat and independently by S. Kurepa [14], [15]).

[6] $\varphi(x)\equiv 0$ is the only solution of equation (17) if we admit also $x=0$.

(a, b) is that the interval (a, b) form a continuous group with respect to the operation $u \circ v = F(u, v)$). If equation (11) *has a continuous and strictly monotonic solution $\varphi_0(x)$, then the function*

$$\varphi(x) = \varphi_0(cx)$$

is its most general solution bounded from one side on a set of a positive measure.

Similar facts can also be proved for equation (12), which may be regarded as a generalization of the Jensen equation (Aczél [9], [15]). *A necessary and sufficiet condition of the existence of a continuous and strictly monotonic solution $\varphi_0(x)$ of equation* (12) *is that the function $F(u, v)$ be continuous and strictly monotonic with respect to each variable and fulfil the condition*

$$F[F(u, v), w] = F[F(u, w), F(w, v)].$$

The function $\varphi(x) = \varphi_0(ax+b)$ is then the most general solution of equation (12) *bounded from one side on a set of a positive measure.*

Equation (12) as well as the equation

$$\varphi(x-y) = F[\varphi(x), \varphi(y)]$$

(Aczél [13]) can easily be reduced to equation (11).

Equation (11) is often called *an addition formula.* Depending on the form of the function $F(u, v)$ we speak about a polynomial, rational, algebraic etc. addition formula. Linear functions and linear functions of the exponential function are the only functions with a polynomial additivity. The functions

$$\varphi(x) = \frac{Ax+B}{Cx+D} \quad \text{and} \quad \varphi(x) = \frac{Ae^{cx}+B}{Ce^{cx}+D}$$

are characterized by a rational additivity. Lastly, any analytic function with an algebraic addition formula is a rational function of x, or a rational function of e^x, or a doubly periodic function (a rational function of the Weierstrassian function $\wp$) (Aczél [19]).

Still more general equation (13), which contains equations (11) and (12) as particular cases, is studied in J. Aczél's monograph [19]. The particular equation

$$(19) \qquad \varphi(ax+by+c) = A\varphi(x) + B\varphi(y) + C \qquad (a \neq 0,\ b \neq 0,\ a+b \neq 0)$$

(Aczél [9], [15], Marcus [1], Daróczy [1]) possesses measurable and non-constant solutions if and only if $A = a$ and $B = b$. The general measurable solution has then the form $\varphi(x) = px+q$, where the constants p and q depend on a, b, c and C. On the other hand, a non-measurable solution of equation (19) may exist also if $a \neq A$ or $b \neq B$. Z. Daróczy [1] has proved that if the equation

$$\varphi(ax+by) = A\varphi(x) + B\varphi(y)$$

has a non-constant solution and one of the numbers a and A is rational, then necessarily $a = A$ (and analogously for b and B). But if one of the numbers a and A is algebraic, then the other must also be algebraic and it must be a root of the same minimal polynomial. In this case a non-constant solution (of course non-measurable) can actually exist though a and A are not equal. This surprising result shows, on one hand, once more the enormous difference between functional and differential equations, and on the other hand the difficulty that must be overcome when one tries to deduce some facts about the structure of the family of solutions of a functional equation from its outer form.

The equation

$$\varphi[f(x, y)] = F[\varphi(x), \varphi(y)], \tag{20}$$

which contains all the equations discussed in the present section, is treated by J. Aczél [15] (cf. also Mineur [1]). The equation

$$\varphi[f(x_1, \ldots, x_p)] = \varphi(x_1) + \cdots + \varphi(x_p)$$

can be reduced (under suitable conditions) to the equation

$$\varphi[f(x, y)] = \varphi(x) + \varphi(y)$$

(Aczél-Kiesewetter [1]), which is a particular case of equation (20). Some more general functional equations were studied by H. Kiesewetter [2].

6. Examples of ordinary functional equations

The system of functional equations

$$\varphi(x+y) = \frac{\varphi(x) + \{[\psi(x)]^2 - [\varphi(x)]^2\}\,\varphi(y)}{1 - \varphi(x)\,\varphi(y)}, \qquad \psi(x+y) = \frac{\psi(x)\,\psi(y)}{1 - \varphi(x)\,\varphi(y)} \tag{21}$$

occurs in the optics and in the probability theory (Brownian motions). System (21) was studied by several authors (G. Stokes 1860, J. Stirling 1914, R. M. Redheffer [1], J. Mycielski-S. Paszkowski [1]) under the supposition of the measurability or boundedness of the functions φ and ψ. J. Aczél [17] gives the general solution of system (21) bounded from one side on a set of a positive measure, making use of the theory of equation (11), to which system (21) can be reduced.

The equation

$$\varphi\left(\frac{x+y}{2}\right) = \sqrt{\varphi(x)\,\varphi(y)}, \tag{22}$$

which is a particular case of equation (12), was used by N. I. Lobačevski [1] to deduce the formula of the parallelism angle. The general measurable solution of equation (22) is $\varphi(x) = ae^{x/k}$.

A generalization of equation (22) to matrix-valued functions has been dealt with by O. E. Gheorghiu and B. Crstici [1].

Similarly the equations

$$\varphi(x) + \varphi(y) = \varphi\left(xy - \sqrt{1-x^2}\,\sqrt{1-y^2}\right), \tag{23}$$

$$\varphi(x) + \varphi(y) = \varphi\left(xy + \sqrt{x^2-1}\,\sqrt{y^2-1}\right), \tag{24}$$

appear in the non-euclidean geometry (in the problem of determining the distance of two points; cf. Aczél-Varga [1], Aczél [15]). The general measurable solutions of equations (23) and (24) considered in suitably restricted sets[7] are the functions

$$\varphi(x) = k \arccos x \quad \text{and} \quad \varphi(x) = k \operatorname{arc\,cosh} x,$$

respectively. Concerning (23), (24) and related equations cf. Aczél [15], [19], Aczél-Varga [1], Ghermănescu [14], Kiesewetter [3], Gołąb-Losonczi [1], [2].

[7] The only function that satisfies (23) for all $x, y \in [-1,1]$ is $\varphi(x) \equiv 0$.

The problem of an axiomatic introduction of the multiplication of vectors leads to the equation (Aczél [8], [15])

$$\varphi(x+y)+\varphi(x-y)=2\varphi(x)\cos y, \tag{25}$$

which may be easily reduced to the equation

$$\varphi(x)+\varphi(x+y)=\psi(y)\,\varphi\left(x+\frac{1}{2}y\right). \tag{26}$$

Equation (26) was treated by S. Kaczmarz [1], who gave its general measurable solution, however, using strong means of the theory of functions of a real variable. J. Aczél [8], [15] finds the general measurable solution $\varphi(x)=c_1\cos x+c_2\sin x$ of equation (25) in a quite elementary way. Similar equation

$$\varphi(x+y)+\varphi(x-y)-2\varphi(x)=2\psi(x)\,\chi(y)$$

has been dealt with by J. Aczél and E. Vincze [1].

The equation

$$\Delta_h^{n+1}\varphi(x)=0, \qquad x\in(-\infty,+\infty), \qquad h\in(0,+\infty), \tag{27}$$

where the difference operator Δ_h^k is defined inductively:

$$\Delta_h^1 f(x)=f(x+h)-f(x), \quad \Delta_h^{k+1}f(x)=\Delta_h^1(\Delta_h^k f(x)), \; k=1,2,3,\ldots \tag{28}$$

characterizes the polynomials of degree $\leqslant n$ among measurable functions (Angheluţă [1], [2]; cf. also Kurepa [11], Ghermănescu [3], [7]). The more general equation

$$\sum_{i=0}^{n+1} c_i\,\varphi(a_i x+b_i y)=0$$

was used by N. Ghirzoiaşiu [1] in order to characterize conics.

Similarly, the equation (Ionescu [1], Stamate [2])

$$\begin{vmatrix} \varphi(x) & \varphi(x+h) & \cdots & \varphi(x+nh) \\ \varphi(x+h) & \varphi(x+2h) & \cdots & \varphi(x+(n+1)h) \\ \vdots & & & \\ \varphi(x+nh) & \varphi(x+(n+1)h) & \cdots & \varphi(x+2nh) \end{vmatrix}=0$$

characterizes the exponential polynomials of order $\leqslant n$ among, say, continuous functions. (In this connection cf. also Radó [4]).

In the information theory one meets the equation

$$\varphi^{-1}\left[\sum_{j=1}^{n}\sum_{k=1}^{m} p_j q_k\,\varphi(-\log_2 p_j q_k)\right]=\varphi^{-1}\left[\sum_{j=1}^{n} p_j\,\varphi(-\log_2 p_j)\right]+\varphi^{-1}\left[\sum_{k=1}^{m} q_k\,\varphi(-\log_2 q_k)\right],$$

$$\left(\sum_{j=1}^{n} p_j=\sum_{k=1}^{m} q_k=1\right).$$

This and related equations have recently been dealt with by J. Aczél, Z. Daróczy, M. Bajraktarević, T. W. Chaundy and J. B. McLeod in connection with some problems in the theory of means, information theory (a characterization of entropies) and statistical thermodynamics (Aczél-Daróczy [1], [2], Daróczy [3], Bajraktarević [10], Chaundy-McLeod [2]).

A study of two operations

$$x \dot{+} y = \frac{x+y}{1+xy}, \qquad x \frown{+} y = x+y-xy,$$

defined for real numbers, leads to the equation

$$\varphi(x)+\varphi(y)-\varphi(x)\varphi(y)=\varphi\left(\frac{x+y}{1+xy}\right),$$

which was treated by J. Aczél [16]. A generalization to matrix-valued functions was solved by Q. E. Gheorghiu [6]. Operations defined for real numbers were studied from a more general point of view by J. Aczél [1] (this problem leads to equation (58)).

We shall mention here also the equation

$$\varphi(xy)=p(y)\,\varphi(x)+q(y)\,x+r(y), \qquad x, y \neq 0, \tag{29}$$

solved under differentiability conditions by S. Gołąb and S. Łojasiewicz [1] and in an elementary way by J. Aczél [17] (cf. also Vincze [6]). Equation (29), which occurred in the paper by Gołab and Łojasiewicz in connection with a theorem concerning the value Θ in the mean-value theorem, is a generalization of a number of equations related to the theory of means, probability, etc. Its general solution bounded on a set of a positive measure are the functions

$$\varphi(x)=a\log|x|+bx+c, \qquad \varphi(x)=ax\log|x|+bx+c,$$

$$\varphi(x)=a|x|^d+bx+c, \qquad \varphi(x)=a|x|^d\operatorname{sgn}x+bx+c.$$

The equation

$$\varphi[x+y\,\varphi(x)]=\varphi(x)\,\varphi(y) \tag{3}$$

is much more difficult than other equations discussed in the present section, as it has a positive implication index. S. Gołąb and A. Schinzel [1] have found continuous solutions of equation [3] and exhibited some discontinuous (and non-measurable) ones, but the general solution of (3) is not known. Measurable solutions of equation (3) have recently been investigated by C. Gh. Popa [1], but the general measurable solution of (3) is not known either. A similar, more general equation (related to some problems in the theory of geometric objects)

$$\varphi(xy)=s(x)\,x^n\,\varphi\left[y+\varphi^{-1}\left(\frac{m}{s(x)\,x^n}\right)\right],$$

where $s(x)=1$ or $s(x)=\operatorname{sgn}x$, has been solved by S. Gołąb [3], [4] under the condition of a differentiability of $\varphi(x)$.

7. Equations of the trigonometric functions

One of the important applications of functional equations is a functional characterization of various functions. The Weierstrassian functions $\wp$ and σ, Riemann's ς function, Euler's Γ function, Lebesgue's singular function, Gauss' arithmetico-geometrical mean, cyclic functions, theta functions, polynomials, rational, exponential and logarithmic functions, and many others can be characterized by functional equations (Baghi-Chaterjee [1], Ghermănescu [17], [22], Siegel [1], Picard [1], Artin [1], Anastassiadis [7], Schmidt [1], Sierpiński [1], Mohr [1], Myrberg [1], Schmidt [3], Maier-Krätzel [1], Angheluță [1], [2], Ghermănescu [3], [7], Kurepa [11], Kuczma [21], Herman [1], Kuczma [24], [19]; cf. also §§ 18, 19 below). But probably the most extensively studied problem of this sort is that of a functional characterization of the trigonometric functions.

One of the oldest functional equations is equation (2). It was studied by J. d'Alembert and S. D. Poisson under the supposition of the analyticity of the function $\varphi(x)$. A. Cauchy [1] found the general continuous solution of equation (2):

$$\varphi(x)=\cos ax, \quad \varphi(x)=\cosh ax, \quad \varphi(x)\equiv 0. \tag{30}$$

Equation (2) finds an application in the problem of the composition of forces (or, in a mathematical formulation, the addition of vectors), which is perhaps the oldest problem solved with the aid of functional equations. Equation (2) appears also in non-euclidean mechanics and geometry (Schimmack [1], Picard [1], Lalan [1], Straszewicz [1], Aczél [5], Maier [1]). O. E. Gheorghiu [3], [5] considered some generalizations of (2) to matrix-valued functions. Equation (14), which is a generalization of equation (2), is connected with the theory of homology groups. J. Aczél [9], [15] describes several methods of solving equation (14).

Since functions (30) are the continuous solutions of equation (2), the latter as well as the equation

$$\varphi(x+y)\,\varphi(x-y)=[\varphi(x)]^2-[\varphi(y)]^2,$$

whose general continuous solution is given by

$$\varphi(x)=ax, \quad \varphi(x)=c\sin ax, \quad \varphi(x)=c\sinh ax$$

(Vietoris [1]), can be used to characterize the trigonometric and hyperbolic functions.

But the most frequent characterization of the trigonometric functions is that by the system of equations

$$\begin{cases} \varphi(x+y)=\varphi(x)\,\varphi(y)-\psi(x)\,\psi(y), \\ \psi(x+y)=\psi(x)\,\varphi(y)+\varphi(x)\,\psi(y). \end{cases} \tag{31}$$

Th. Angheluță [3] solved system (31) under the assumption of the continuity of the functions φ and ψ. P. Montel [1] studied the more general system

$$\begin{cases} \varphi(x+y)=G[\varphi(x),\ \psi(x),\ \varphi(y),\ \psi(y)], \\ \psi(x+y)=H[\varphi(x),\ \psi(x),\ \varphi(y),\ \psi(y)], \end{cases}$$

(also under continuity conditions). M. Ghermănescu [4] proved that the general real measurable solution of system (31) is given by

$$\varphi(x)=e^{ax}\cos bx, \quad \psi(x)=e^{ax}\sin bx,$$

where a, b are arbitrary constants. He also proved that the general real, linearly measurable solution of the system of equations[8]

$$\begin{cases} \lambda(x,y)\,\varphi(x+y)=\varphi(x)\,\varphi(y)-\psi(x)\,\psi(y), \\ \lambda(x,y)\,\psi(x+y)=\psi(x)\,\varphi(y)+\varphi(x)\,\psi(y), \end{cases} \tag{32}$$

[8] Because of the function $\lambda(x, y)$, the equations in systems (32) and (33) are in fact partial functional equations.

is given by

$$\varphi(x)=A(x)\,e^{ax}\cos bx, \quad \psi(x)=A(x)\,e^{ax}\sin bx, \quad \lambda(x, y)=\frac{A(x)\,A(y)}{A(x+y)},$$

where $A(x)$ is an arbitrary measurable function and a, b are arbitrary constants. The similar system

$$\text{(33)} \qquad \begin{cases} \lambda(x, y)\,\varphi(xy)=\varphi(x)\,\varphi(y)+k\,\psi(x)\,\psi(y), \\ \lambda(x, y)\,\psi(xy)=\psi(x)\,\varphi(y)+\varphi(x)\,\psi(y)-c\,\psi(x)\,\psi(y), \end{cases}$$

was dealt with by O. E. Gheorghiu-V. Mioc-B. Crstici [1].

The equation

$$\text{(34)} \qquad \tau(x+y)=\frac{\tau(x)+\tau(y)}{1-\tau(x)\,\tau(y)}$$

may be reduced to system (32). The general measurable solution of equation (34) is $\tau(x)=\tan bx$.

All the real solutions of equations (31) have been determined by L. Vietoris [1], all the complex solutions of equations (31), (35) and of some more general equations have been found by E. Vincze [5], [6]. In the complex domain the continuous solutions of the system consisting of equations (31) and

$$[\varphi(x)]^2+[\psi(x)]^2=[\varepsilon(x)]^2, \quad \varepsilon(x+y)=\varepsilon(x)\,\varepsilon(y), \qquad \varepsilon(x)\neq 0,$$

have been given by J. Aczél [17]. O. Hájek [1] found the solutions holomorphic in a neighbourhood of the origin to the equation

$$\varphi(x+y)=a\,\varphi(x)\,\varphi(y)+b\,\psi(x)\,\psi(y)$$

as well as to the equation

$$\varphi(x+y)=a\,\psi(x)\,\varphi(y)+b\,\varphi(x)\,\psi(y).$$

The system of equation

$$\text{(35)} \qquad \begin{cases} \varphi(x+y)=\varphi(x)\,\varphi(y)+\psi(x)\,\psi(y), \\ \psi(x+y)=\psi(x)\,\varphi(y)+\varphi(x)\,\psi(y), \end{cases}$$

and (35) together with

$$[\varphi(x)]^2-[\psi(x)]^2=[\varepsilon(x)]^2, \quad \varepsilon(x+y)=\varepsilon(x)\,\varepsilon(y), \qquad \varepsilon(x)\neq 0,$$

(which characterize the hyperbolic functions) have also been treated by numerous authors (e.g. Angheluţă [3], Vietoris [1], Aczél [17]; regarding the ample bibliography of functional equations of the trigonometric and related functions cf. Aczél [19], Vincze [3], [5]).

H. E. Vaughan [1] considers the single equation (cf. also Vincze [6])

$$\text{(36)} \qquad \varphi(x-y)=\varphi(x)\,\varphi(y)+\psi(x)\,\psi(y).$$

He proves that $0\leq\varphi(0)\leq 1$. If $\varphi(0)=0$, then $\varphi(x)\equiv 0$ and $\psi(x)\equiv 0$. If $0<\varphi(0)=c<1$, then equation (36) possesses exactly two solutions:

$$\varphi(x)\equiv c, \quad \psi(x)\equiv\sqrt{c-c^2} \quad \text{and} \quad \varphi(x)\equiv c, \quad \psi(x)\equiv-\sqrt{c-c^2}$$

Lastly, if $\varphi(0)=1$ and $\varphi(x)$ or $\psi(x)$ is continuous at least at one point, then

$$\varphi(x)=\cos ax, \quad \psi(x)=\sin ax.$$

In particular, if functions $\varphi(x)$ and $\psi(x)$ satisfy equation (36) and $\lim_{h\to 0}\frac{\psi(h)}{h}=1$, then $\varphi(x)=\cos x$, $\psi(x)=\sin x$. But in the case where $\varphi(0)=1$ equation (36) has also totally discontinuous solutions.

A number of analogous equations in abstract spaces (Hilbert spaces, Banach spaces etc.) have been treated by S. Kurepa [2], [3], [5], [6], [7], [8], [9], [10], [12], [13], [14], [15], [16], D. Ž. Đoković [5], G. Maltese [1], F. Vajzović [1], M. Kuczma [17].

8. Ordinary functional equations with several unknown functions

One of the striking features of functional equations is the fact that, contrary to differential equations, a single equation can determine more than one function (e.g. equation (36)). Here first of all the Pexider equations (Pexider [1])

$$\alpha(x+y)=\beta(x)+\gamma(y), \tag{37}$$

$$\alpha(x+y)=\beta(x)\,\gamma(y), \tag{38}$$

$$\alpha(xy)=\beta(x)+\gamma(y), \tag{39}$$

$$\alpha(xy)=\beta(x)\,\gamma(y), \tag{40}$$

must be mentioned. Equations (37)—(40) are an immediate generalization of the Cauchy equations, to which they can be easily reduced. The general solutions of the Pexider equations are of the form

(37′), (39′) $$\alpha(x)=\varphi(x)+a+b,\quad \beta(x)=\varphi(x)+a,\quad \gamma(x)=\varphi(x)+b,$$

(38′), (40′) $$\alpha(x)=ab\,\varphi(x),\quad \beta(x)=a\,\varphi(x),\quad \gamma(x)=b\,\varphi(x),$$

where a, b are arbitrary constants and $\varphi(x)$ is an arbitrary solution of the corresponding Cauchy equation (1), (16), (17), (18), (Vincze [6], [8]). Taking as $\varphi(x)$ in formulae (37′)—(40′) measurable solutions of the corresponding Cauchy equation (formulae (9), (16′), (17′), (18′)) one can obtain the general measurable solution of equations (37)—(40).

Concerning further equations of this sort the reader is referred to Aczél [19], [26], Hosszú [11], Stamate [3], Vincze [6], [8].

E. Vincze [3], [6], considered the equation

$$\varphi(x+y)=\alpha(x)\,\beta(y)+\gamma(x)\,\delta(y) \tag{41}$$

for complex-valued functions φ, α, β, γ, δ of a complex variable. In (41) the variables are supposed to range over an additive group of complex numbers. Equation (41) contains the equations of the trigonometric and hyperbolic functions as well as the Pexider equations as particular cases.

The general solution of the equation

$$\varphi(x+y)=\alpha(x)+\beta(y)+\gamma(x)\,\delta(y), \tag{42}$$

which contains that considered by I. Stamate [1] as a particular case, was given by Z. Daróczy [2], E. Vincze [6].

The still more general equation

$$\varphi(x+y)=\sum_{i=1}^{n}\psi_i(x)\,\chi_i(y), \tag{43}$$

containing both (41) and (42), was solved under differentiability conditions by T. Levi-Civita [1] and via distribution theory by I. Fenyö [1]. The family of continuous functions φ which satisfy equation (43) together with continuous functions ψ_i, χ_i coincides with the family of solutions of linear differential equations of order n with constant coeficients (Radó [4]).

An akin system of functional equations

$$\varphi_i(x+y)=\sum_{j,\,k=1}^{n}c_{ijk}\,\varphi_j(x)\,\varphi_k(y),\quad i=1,\ldots,n, \tag{44}$$

has recently been solved under differentiability conditions by W. Eichhorn [1]. System (44) may be regarded as a generalization of systems (31) and (35) as well as of addition formulae of cyclic and circular functions (Schmidt [2], [3]).

In the case of equations (37)—(42), in order to obtain the general solution, the authors reduce the equation in question to a suitable Cauchy equation, whose general solution is well known (cf. §4). Then, in order to obtain a solution of a rather simple form, it is enough to postulate e.g. the measurability of the sought functions. But, since the Cauchy equations have non-measurable solutions, the same is valid for equations (37)—(42) as well as for the more general equation (43). To construct such a non-measurable solution, one must use the axiom of choice.

The situation is different in the case of the equation

$$\sum_{i=1}^{n}\psi_i(x)\,\chi_i(y)=0, \tag{45}$$

which is somewhat similar to equation (43), but is of order zero, whereas (43) has order 1. Equation (45) can be solved in a quite elementary way without any suppositions whatever about the sought functions ψ_i, χ_i (Aczél [20]).

The more general equation

$$\sum_{i=1}^{n}\psi_i^1(x_1)\,\psi_i^2(x_2)\ldots\psi_i^k(x_k)=0$$

has been solved without any suppositions about the sought functions ψ_i^j by L. Losonczi [1].

Some particular cases of equation (45) was previously dealt with by D. S. Mitrinović [1], J. Aczél [17], O. E. Gheorghiu [4] and T. Popoviciu [1].

F. Radó [1] introduced the equation

$$\begin{vmatrix}\varphi(x) & \varphi(x+h) & \varphi(x+2h)\\ \psi(x) & \psi(x+h) & \psi(x+2h)\\ \chi(x) & \chi(x+h) & \chi(x+2h)\end{vmatrix}=0$$

as a condition for the linear dependence of three functions. A more general equation was dealt with by T. Popoviciu [1] (cf. also Kiesewetter [2], Vincze [6]).

One is led to functional equations with several unknown functions[9] also by the problem of a representation of functions of several variables as superpositions of functions of a smaller number of variables. In this connection we mention here only the excellent recent papers by A. N. Kolmogorov [2], [3], [4] and V. I. Arnold [1], [2], [3], [4], [5]. Further refernces may be found in the monograph Aczél [19].

9. Matrix equations

Many of the partial functional equations may be written in a simpler form with a use of a vector or matrix notation. An important example of such an equation (and rather a system of equations) is

$$\Phi(X \cdot Y) = \Phi(X) \cdot \Phi(Y), \tag{46}$$

where X, Y are $n \times n$ matrices, Φ is an $m \times m$ matrix-function and the dot denotes the multiplication of matrices.

Equation (46) plays an important part in the invariants theory (Perron [1]), axiomatic definition of a determinant (Bergman [1], Gáspár [1], Kurepa [17]) and in the theory of geometric objects (Aczél-Gołąb [1], Kucharzewski-Kuczma [5]). It was solved for various values of m and n, under very strong regularity suppositions regarding the function Φ, by O. Perron [1], P. Reisch [1] and I. Schur [1], [2], [3]. Measurable solutions of equation (46) as well as of related equations

$$\Phi(X \cdot Y) = \Phi(X) + \Phi(Y), \tag{47}$$

$$\Phi(X + Y) = \Phi(X) \cdot \Phi(Y), \tag{48}$$

$$\Phi(X + Y) = \Phi(X) + \Phi(Y), \tag{49}$$

have recently been given by A. Kuwagaki [2] (cf. also Gheorghiu [2]). S. Kurepa [4] has solved equations (46)—(49) for arbitrary m, n, without any regularity suppositions. Instead, he assumes that Φ fulfils the invariance condition

$$\Phi(U^{-1} \cdot X \cdot U) = \Phi(X)$$

for all $n \times n$ matrices X and for all matrices U belonging to a certain class (e.g. the class of unitary, orthogonal or non-singular matrices), which causes that he has not obtained all the solutions. In fact S. Kurepa [4] has studied the more general equations

$$\Phi(X \cdot Y) = F[\Psi(X) \cdot \Psi(Y),\ \Theta(X) + \Theta(Y)], \tag{50}$$

$$\Phi(X + Y) = F[\Psi(X) \cdot \Psi(Y),\ \Theta(X) + \Theta(Y)], \tag{51}$$

and has proved that (under suitable invariance conditions) a function $\Phi(X)$ satisfying equation (50) resp. (51) depends only on the determinant resp. trace of the matrix X.

All the invertible solutions of equation (46) ($m = n$) have been found by J. Dieudonné [1] and A. Zajtz [2]. The case where $n \geqslant m$ has also been treated by A. Zajtz, who reduced it to the case $m = 1$.

Without any suppositions whatever equation (46) has been solved in the case $m = 1$, $n = 2$ by S. Gołąb [2]. His result reads as follows:

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[9] These are ordinary (of a positive implication index) as well as partial functional equations.

The general solution of equation (46) $(m=1,\ n=2)$ *is of the form*

$$\Phi(X)=\varphi(\det X),$$

where $\varphi(u)$ *is an arbitrary scalar-valued function of a single variable satisfying equation* (18).

S. Gołąb's theorem has been proved in the case $m=1$, n arbitrary, by M. Kucharzewski [1] and then also by M. Hosszú [7]. A. Zajtz [1] has proved the following, still more general theorem:

If a scalar valued function $\Phi(X)$ *of a matrix argument satisfies the equation*

$$\Phi(X\cdot Y)=H[\Phi(X),\ \Phi(Y)],$$

where $H(u,v)=H(v,u)$ *is a symmetric function, then*

$$\Phi(X)=\varphi(\det X),$$

where $\varphi(u)$ *is a scalar-valued function of a scalar argument satisfying the equation*

$$\varphi(xy)=H[\varphi(x),\ \varphi(y)].$$

A. Zajtz has obtained this result as a consequence of a more general one saying that *a function* $\Phi(X)$ (*of a matrix argument and taking values in an arbitrary set*) *such that*

$$\Phi(X\cdot Y\cdot Z)=\Phi(X\cdot Z\cdot Y)$$

for arbitrary $n\times n$ *matrices* X, Y, Z, *depends only on the determinant of* X.

In the case $m=n=2$ equation (46) has been solved without any suppositions whatever by M. Kucharzewski and M. Kuczma [3]. The solution is given by

$$\Phi(X)=\begin{Vmatrix}\varphi(\det X) & 0\\ 0 & \varphi(\det X)\end{Vmatrix}\cdot C\cdot X\cdot C^{-1},\quad \Phi(X)=\Lambda(\det X),$$

where $\varphi(u)$ is an arbitrary function (scalar-valued and of a scalar argument) satisfying equation (18), C is an arbitrary non-singular 2×2 matrix, and $\Lambda(u)$ is a matrix function of a scalar argument satisfying the equation[10]

$$\Lambda(xy)=\Lambda(x)\cdot\Lambda(y). \tag{52}$$

The general solution or general measurable solution of equation (52) for 2×2 matrices Λ has been given by O. E. Gheorghiu [1], A. Balogh [1], M. Kucharzewski-M. Kuczma [1], (cf. also Ghermănescu [6]). For 3×3 matrices Λ equation (52) has been completely solved by M. Kuczma and A. Zajtz [1].

In the theory of geometric objects an important part is played by the system consisting of equation (46) and the equation

$$\Psi(X\cdot Y)=\Phi(X)\cdot\Psi(Y)+\Psi(X), \tag{53}$$

where Ψ is an $m\times 1$ matrix-function. For $m=1$, n arbitrary the system of equations (46), (53) has been solved by M. Kuczma [4], and for $m=n=2$ by M. Kucharzewski and M. Kuczma [4]. A similar system has recently been dealt with by J. Aczél [21] in connection with an algebraic problem.

Let us note, however, that the problem of finding *all solutions* of equation (46) and of system (46), (53), *without any suppositions whatever regarding the functions* Φ, Ψ, *for arbitrary* m *and* n, *still remains unsolved.*

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[10] Equation (52) is thus an ordinary functional equation (and rather a system of ordinary functional equations).

10. The equation of translation

Functional equations are one of the main tools in the theory of geometric objects (Aczél-Gołąb [1], Kucharzewski-Kuczma [5]). One of the principal problems in that theory, the classification problem, leads to the functional equation

$$\Phi[\Omega; T_3] = \Phi\{\Phi[\Omega; T_1]; T_2\}. \tag{54}$$

Here Ω denotes a quantity (or a system of quantities) called the components of a geometric object and playing in (54) the rôle of variables. T_1 and T_2 are systems of parameters characterizing transformations $\mathcal{T}_1$ and $\mathcal{T}_2$ of the coordinates and T_3 is the system of parameters characterizing the superposition of the transformations $\mathcal{T}_1$ and $\mathcal{T}_2$. The parameters T_1 and T_2 also play the rôle of variables, while T_3 are expressed in terms of T_1 and T_2. Equation (54) in quite general case has not yet been solved.

Also further problems of the theory of geometric objects (equivalence of objects, determination of concomitants and algebras of objects) can be reduced to that of solving a suitable functional equation. Presenting these problems and results would take too much place. The reader is referred to the book of Aczél and Gołąb [1] or to the expository article Kucharzewski-Kuczma [5], which both contain also an extensive bibliography.

The same equation (54) may have also a number of further interpretations and consequently appears also in other domains. As the equation of one-parameter translation

$$\varphi[\varphi(x, u), v] = \varphi(x, u+v) \tag{55}$$

it was treated, under various conditions, in the papers Aczél [4], Aczél-Kalmár-Mikusiński [1], Prešić [1], Hosszú [8], [10], [12]. In the case where φ and x are n-dimensional vectors, equation (55) was discussed in the paper Aczél [11]. Also other interpretations of equation (55) are mentioned there: it is satisfied by the integrals of the differential equation of the stationary motion, as well as by the generating function of the chain reaction in the homogeneous case. In the inhomogeneous case one obtains the equation

$$\varphi[\varphi(x, s, t), t, u] = \varphi(x, s, u),$$

which is also contained in the general form (54). For multi-parameter transformations equation (54) was considered by J. Aczél and M. Hosszú [1].

In the one-dimensional case *the general continuous solution of equation* (55) *has the form*

$$\varphi(x, u) = g^{-1}[g(x) + u], \tag{56}$$

where $g(x)$ is an arbitrary continuous and strictly increasing function (Aczél-Kalmár-Mikusiński [1]). This result has been generalized to the case where the variables u, v in (55) are in an abelian group or groupoid by S. Prešić [1] and M. Hosszú [8].

Equation (55) occurs also in the iteration theory. The natural iterates of a function $f(x)$ are defined by the relations

$$f^1(x) = f(x), \quad f^{k+1}(x) = f[f^k(x)]. \tag{57}$$

For integral values of u, v the function $\varphi(x, u) = f^u(x)$ satisfies equation (55). So solutions of (55) may be regarded as an extension of the notion of an iterate to arbitrary real iteration indices. Condition (57) leads then to

$$g^{-1}[g(x)+1] = f(x),$$

which means that the function $g(x)$ in (56) must satisfy the Abel equation

$$g[f(x)] = g(x)+1$$

(cf. § 16). The iteration theory was studied among others by E. Schröder [1], M. Ward-F. B. Fuller [1], J. Hadamard [1], M. Töpfer [1], M. Bajraktarević [3], S. Łojasiewicz [2], J. Aczél [10], M. K. Fort [1], G. Szekeres [1], [2], [3], I. N. Baker [1], [2], [5], P. Erdös-E. Jabotinsky [1], H. Michel [1], L. Berg [2], [3], A. Lundberg [1], B. Muckenhoupt [1], M. Kuczma [18], [20], M. A. McKiernan [3]. Cf. also § 17.

11. Equations of algebraic structures

In the algebra the problem of determining the most general operations fulfilling certain conditions leads to a number of important functional equations:

(58) $$\varphi[\varphi(x, y), z] = \varphi[x, \varphi(y, z)] \qquad \text{(associativity)},$$

(59) $$\begin{cases} \varphi[x, \varphi(y, z)] = \varphi[\varphi(x, y), \varphi(x, z)] \\ \varphi[\varphi(x, y), z] = \varphi[\varphi(x, z), \varphi(y, z)] \end{cases} \qquad \text{(autodistributivity)},$$

(60) $$\varphi[\varphi(x, y), \varphi(u, v)] = \varphi[\varphi(x, u), \varphi(y, v)] \qquad \text{(bisymmetry)},$$

(61) $$\varphi(x, y) = \varphi[\varphi(x, u), \varphi(y, u)] \qquad \text{(transitivity)},$$

etc. These equations as well as various their generalizations have been dealt with by many authors, starting with N. H. Abel [1], then in a series of papers of A. R. Schweitzer, L. Brouwer, T. Farago, J. G. Mikusiński, S. Gołąb, B. Knaster, C. Ryll-Nardzewski, A. Kuwagaki, A. Sade, J. Aczél, and M. Hosszú, to name but a few. Among more recent papers we mention here Maier [1], Stein [1], Belousov [1], Radó [2], [3], Aczél [1], [17], Aczél-Belousov-Hosszú [1], Hosszú [1], [2], [3], [4], [6], [12], Ghermănescu [20], Sade [1], [2]. A detailed discussion of those results would be impossible. The reader is referred to the monograph Aczél [19], where also a more accurate bibliography can be found. A treatment of equations of this kind constitutes a great part of J. Aczél's book. Here, as an example, we shall quote only a result of M. Hosszú [2].

A continuous function $\varphi(x, y)$, *strictly monotonic with respect to* y, *satisfies equation* (61) *if and only if there exists a continuous and strictly monotonic function* $f(z)$ *such that*

$$\varphi(x, y) = f^{-1}[f(x) - f(y)].$$

The solutions of equations (58)—(60) are given (under suitable assumptions) by:

$$\varphi(x, y) = f^{-1}[f(x) + f(y)]$$

for the equation of associativity and

$$\varphi(x, y) = f^{-1}[(1-q) f(x) + q f(y)]$$

for the equations of bisymmetry and autodistributivity.

Many of the papers mentioned above discuss generalizations of equations (58)—(61) in which several unknown functions occur. E.g. J. Aczél, V. D. Belousov and M. Hosszú [1] treat the equations

$$\varphi[\alpha(x, y), z]=\psi[x, \beta(y, z)]$$

(a generalization of (58)) and

$$\varphi[\alpha(x, y), \beta(u, v)]=\psi[\gamma(x, u), \delta(y, v)]$$

(a generalization of (60)), of which the first contains 4 and the second 6 unknown functions.

The most general pair of binary operations $\varphi(x, y)$, $\psi(x, y)$, of which the first is associative (i.e. satisfies (58)) and the second is distributive with respect to the first:

$$\psi[\varphi(x, y), z]=\varphi[\psi(x, z), \psi(y, z)]$$

(so $\varphi(x, y)$ and $\psi(x, y)$ generalize the ordinary addition and multiplication) is given by

$$\varphi(x, y)=f^{-1}[f(x)+f(y)], \qquad \psi(x, y)=f^{-1}[f(x) g(y)],$$

provided $\varphi(x, y)$ is continuous and strictly increasing and $\psi(x, y)$ is bounded from below.

These and similar equations occur in the axiomatic foundations of the probability theory (Aczél [12], [23]). The equation of associativity has been used by B. Schweizer and A. Sklar [1] in their investigations of the triangle inequalities in statistical metric spaces.

The functional equation of associativity for n-place functions has been investigated by E. Vincze [1], [4], M. Hosszú [15].

12. Further examples of partial functional equations

Much simpler than the equations discussed in the preceding section is the Sinzow equation (Sinzow [1])

$$\varphi(x, y)+\varphi(y, z)=\varphi(x, z), \tag{62}$$

whose general solution is given by

$$\varphi(x, y)=f(y)-f(x),$$

where $f(x)$ is an arbitrary function. Equation (62) has been studied by several authors (in particular, S. Gołąb [1] and P. Rossier [1] have applied the Sinzow equation in the non-euclidean geometry).

The equation

$$\Pi(s, u)=\Pi(s, t) \cdot \Pi(t, u),$$

where Π is an $n \times n$ matrix, may be regarded as a generalization of equation (62). This and similar equations find applications in the probability theory (Fréchet [1], Aczél [7], [10], [15], [17]).

Functional equations occur also in the theory of means (cf. e.g. Kolmogorov [1], Aczél [14], [17], Aczél-Daróczy [2], Hosszú [4], Bajraktarević [2],

[6], [10]). E. Rufener [1] applies the theory of quasiarithmetic means to survival functions. M. Hosszú and E. Vincze [2] apply the theory of means to a problem from the probability theory. Some problems of the statistical thermodynamics lead to functional equations (Chaundy-McLeod [1], [2]).

Investigations of invariants also are connected with functional equations (cf. e.g. Jabotinsky [1], Perron [1], Kurepa [1]). We shall mention here a theorem from the paper Aczél-Gołąb-Kuczma-Siwek [1] to the effect that a homographic invariant of four points of the projective line must be a function of the anharmonic ratio of these points. So the anharmonic ratio can be defined with the aid of the functional equation

$$\Phi\left(\frac{ax_1+b}{cx_1+d},\ \frac{ax_2+b}{cx_2+d},\ \frac{ax_3+b}{cx_3+d},\ \frac{ax_4+b}{cx_4+d}\right)=\Phi(x_1,\ x_2,\ x_3,\ x_4).$$

R. M. Redheffer [2] considered a function $\mu(x, y)$ satisfying the differential equation

$$\frac{\partial \mu}{\partial y}=a(y)+2\,b(y)\,\mu(x,y)+c(y)\,[\mu(x,y)]^2$$

and certain related functions $\nu(x, y)$ and $\omega(x, y)$. These functions satisfy the system of functional equations

$$\frac{\mu(x,z)-\mu(y,z)}{\mu(x,y)}=\exp[-\nu(x,y)+\nu(x,z)+\nu(y,z)],$$

$$1-\mu(x,y)\,\omega(y,z)=\exp[\ \nu(x,y)-\nu(x,z)+\nu(y,z)],$$

$$\frac{\omega(x,z)-\omega(x,y)}{\omega(y,z)}=\exp[\ \nu(x,y)+\nu(x,z)-\nu(y,z)],$$

which is connected with a class of problems occurring in the theory of electromagnetism.

The equation of homogeneous functions

$$\varphi(tx, ty)=t^k\,\varphi(x,y) \tag{63}$$

was solved already by L. Euler 1755 (cf. Aczél [19]). Since then a number of people have studied various generalizations of equation (63). So e.g. V. Alaci [1], [2], [3] has considered (under differentiablity conditions) the equations

$$\varphi[\psi_1(t)\,x,\ \psi_2(t)\,y,\ \psi_3(t)\,z]=\Psi(t)\,\varphi(x,y,z) \tag{64}$$

and

$$\varphi[x+\psi_1(t),\ y+\psi_2(t),\ z+\psi_3(t)]=\Psi(t)\,\varphi(x,y,z).$$

Particular cases of equation (64)

$$\varphi(tx,ty,tz)=\varphi(x,y,z),\ \ \varphi(tx,ty,z)=\psi(t)\,\varphi(x,y,z),\ \ \varphi(tx,y,tz)=t\,\varphi(x,y,z), \tag{65}$$

have been dealt with by E. Vincze [2] in connection with an economic problem. The solution of system (65) is given by

$$\varphi(x,y,z)=\frac{x}{x_0}\cdot\frac{y_0}{y}\cdot\left(\frac{x}{x_0}\cdot\frac{z_0}{z}\right)^{\alpha},$$

were x_0, y_0, z_0 and α are constants which must be determined from the experimental data. Similar, more general equations, have been studied by J. Aczél [17] and M. Hosszú-E. Vincze [1]. The quite general equation

$$\varphi[f_1(x_1,\ldots,x_n,t_1,\ldots,t_p),\ldots,f_n(x_1,\ldots,x_n,t_1,\ldots,t_p)] \\ = F[\varphi(x_1,\ldots,x_n),t_1,\ldots,t_p]$$

has recently been investigated by S. Topa [1].

13. Cyclic equations

There is an important class of functional equations which can be specialized to equations of rank $p \geqslant 2$ as well as to those of rank 1. These are equations of the form e.g. (cf. Mitrinović-Đoković [11])

$$\sum_{i=0}^{n-1} A_i(x)\,\varphi(P^i x) = B(x), \tag{66}$$

where $x=(x_1,\ldots,x_p)$ and P is an operator of period n, i.e. $P^n=P$. In the case $p=1$ (66) is an equation with an n-periodic argument of the sort treated by M. Ghermănescu [8], [11] (cf. § 20). In the case $A_i(x)\equiv 1$, $B(x)\equiv 0$, the most general solution of equation (66) is (cf. Aczél-Ghermănscu-Hosszú [1])

$$\varphi(x)=F(x)-F(Px),$$

where $F(x)$ is an arbitrary function. Here the values of φ may lie in an arbitrary module (an additive abelian group) in which the following condition is fulfilled:

($\mathcal{A}_n$) every equation $n\xi=\alpha$ has a unique solution ξ.

The case where the iterates P^i of the operator P form a finite group of order n has been treated by S. B. Prešić [4], [5] (cf. also Mitrinović [8]).

An important case is that of a cyclic operator. The cyclic operator C_n is defined by

$$C_n F(x_1,\ldots,x_n)=F(x_2,\ldots,x_n,x_1).$$

Here F need not depend on all the variables $x_1,\ldots,x_n$. E.g. $C_3 F(x_1,x_2)=$ $=F(x_2,x_3)$. The operator C_n is, of course, n-periodic.

The equation

$$\sum_{i=1}^{n} C_n^{i-1}\,\varphi(x_1,\ldots,x_p)=0, \qquad p\leqslant n, \tag{67}$$

(which may be regarded as a particular case of (66)), where the values of φ lie in a module in which condition ($\mathcal{A}_m$) is fulfilled for every $m\leqslant n$, has been solved by Aczél-Ghermănescu-Hosszú [1]. (More general linear cyclic equations have been dealt with in Hosszú [9]). In the case where $n\geqslant 2p-1$ it is enough to assume that ($\mathcal{A}_n$) is fulfilled. The solution has then the form

$$\varphi(x_1,\ldots,x_p)=F(x_1,\ldots,x_{p-1})-F(x_2,\ldots,x_p),$$

where F is an arbitrary function. If we drop the condition ($\mathcal{A}_n$), then the solution of (67) is (Đoković [6])

$$\varphi(x_1,\ldots,x_p)=F(x_1,\ldots,x_{p-1})-F(x_2,\ldots,x_p)+\theta,$$

where θ is an arbitrary solution of the equation $n\theta=0$. The case $p<n<2p-1$ is much more difficult. If $(\mathcal{A}_m)$ is fulfilled for $m\leqslant n$, then the solution of equation (67) is

$$\varphi(x_1,\ldots,x_p)=G_0(x_1,\ldots,x_{p-1})-G_0(x_2,\ldots,x_p)+$$
$$+\sum_{k=1}^{[(2p-n)/2]}[G_k(x_1,\ldots,x_k,\ x_{n-p+k+1},\ldots,x_p)-$$
$$-G_k(x_{p-k+1},\ldots,x_p,\ x_1,\ldots,x_{2p-n-k})],$$

where G_k are arbitrary functions. In the general case (without $\mathcal{A}$-conditions) the solution is not known.

A somewhat more general equation

$$\sum_{i=1}^{n} C_n^{i-1}\varphi_i(x_1,\ldots,x_p)=0$$

is solved under the assumption that $n\geqslant 2p-1$ in Đoković [6]. Some particular cases of this equation have been solved by D. S. Mitrinović [5], [7] in the more difficult case where $p<n<2p-1$.

In the above examples the variables have been only permuted. But one can also apply some operations of another sort. Such is e.g. the equation (Mitrinović-Đoković [1])

$$\sum_{i=1}^{n+1} C_{n+1}^{i-1}\varphi(x_1,x_2,\ldots,x_{n-1},x_n\circ x_{n+1})=0, \tag{68}$$

where x_i belong to a semigroup $\mathfrak{G}$ with a unity, $\circ$ denotes the operation in $\mathfrak{G}$ and the values of φ lie in a module fulfilling condition $(\mathcal{A}_{n+1})$. Equation (68) is fulfilled by

$$\varphi(x_1,\ldots,x_n)=g_1(x_1,x_2,\ldots,x_{n-1}\circ x_n)-g_1(x_2,x_3,\ldots,x_n\circ x_1)+ \tag{69}$$
$$+\sum_{j=2}^{[(n+1)/2]}[g_j(x_1,\ldots,x_{n-j}\circ x_{n-j+1},\ldots,x_n)-$$
$$-g_j(x_{j+1},\ldots,x_n,x_1,\ldots,x_{j-1}\circ x_j)],$$

where g_j are arbitrary functions. D. S. Mitrinović and D. Ž. Đoković conjecture that *for odd n function* (69) *is the general solution of equation* (68) (Mitrinović-Đoković [10]). This conjecture has been proved for $n=1,3,5,7$ (Đoković [1], [5]). For even n (69) is not the general solution of (68).

Similar equations

$$\sum_{i=1}^{m+n} C_{m+n}^{i-1}\varphi_i(x_1+\cdots+x_m,\ x_{m+1}+\cdots+x_{m+n})=0,$$

$$\sum_{i=1}^{m+n+p} C_{m+n+p}^{i-1}\varphi(x_1+\cdots+x_m,\ x_{m+1}+\cdots+x_{m+n},\ x_{m+n+1}+\cdots+x_{m+n+p})=0,$$

have been solved under the supposition of the continuity of the functions φ by S. Prešić-D. Ž. Đoković [1] and D. Ž. Đoković [2], respectively.

The equation

$$\sum_{i=1}^{n} \varphi(x_i + \cdots + x_{i+k_i-1}, x_{i+k_i} + \cdots + x_{i+n-1}) = 0$$

(where $x_{i+n} \equiv x_i$) is connected with a cyclic matrix (Mitrinović-Đoković [8], [9]).

Another example of a cyclic equation is

$$(70)\quad \sum_{i=1}^{n} [C_n^{i-1}\, \varphi(x_1 \circ x_2, x_3) - C_n^{i-1}\, \varphi(x_1 \vee x_2, x_3) - C_n^{i-1} \varphi(x_1 \circ x_2, x_3 \vee x_4)] = 0,$$

where the variables x_i belong to a set E endowed with two inner operations $\circ$ and $\vee$, with an element e such that $x \vee e = e \vee x = e$ and $x \circ e = e \circ x = x$ for every $x \in E$, and the values of φ lie in a module fulfilling $(\mathcal{A}_n)$. Under these hypotheses the general solution of equation (70) is given by

$$(71)\qquad \varphi(x_1, x_2) = F(x_1) - F(x_2),$$

where F is an arbitrary function, *provided that* $n \geqslant 5$. In the case where $n = 4$, in order to prove that (71) is the general solution of equation (70) one must assume additionally that the operation $\vee$ is associative, or commutative, or it has a unity u (cf. Mitrinović-Đoković [3], [5], [10], where also some generalizations are considered). It is an open problem whether this additional hypothesis is in fact necessary.

In the theory of cyclic functional equations a rôle is played by the operator

$$S_{n-1}^{x_1, \ldots, x_n} f(t_1, \ldots, t_{n-1}) = (-1)^{n-1} f(x_1, x_2, \ldots, x_{n-1}) - f(x_2, x_3, \ldots, x_n) + \\ + \sum_{k=1}^{n-1} (-1)^{k+1} f(x_1, x_2, \ldots, x_k + x_{k+1}, x_{k+2}, \ldots, x_{n-1}, x_n)$$

(Mitrinović-Đoković [2], [4], [7], [10]). D. S. Mitrinović and D. Ž. Đoković [4] have proved that *the general differentiable solution of the equation*

$$(72)\qquad S_n^{x_1, \ldots, x_{n+1}} \varphi(t_1, \ldots, t_n) = 0$$

is given by

$$\varphi(x_1, \ldots, x_n) = S_{n-1}^{x_1, \ldots, x_n} F(t_1, \ldots, t_{n-1}),$$

where $F(x_1, \ldots, x_{n-1})$ *is an arbitrary differentiable function.*

For $n = 2, 3, 4$ the above result had previously been proved by S. Kurepa [1]. J. Erdös [1] proved that the function

$$(73)\qquad \varphi(x_1, x_2) = S_1^{x_1, x_2} F(t) = F(x_1 + x_2) - F(x_1) - F(x_2)$$

(with an arbitrary continuous F) is the general continuous solution of the equation

$$(74)\quad S_2^{x_1, x_2, x_3} \varphi(t_1, t_2) \equiv \varphi(x_1, x_2) - \varphi(x_2, x_3) + \varphi(x_1 + x_2, x_3) - \varphi(x_1, x_2 + x_3) = 0.$$

He has also proved (cf. Mitrinović-Đoković [10] and also Hosszú [13]) that the general solution of equation (74) has the form

$$\varphi(x_1, x_2) = F(x_1 + x_2) - F(x_1) - F(x_2) + G(x_1, x_2),$$

where $F(x)$ is an arbitrary function and $G(x_1, x_2)$ is an arbitrary function satisfying the conditions

$$G(x_1, x_2) = -G(x_2, x_1), \quad G(x_1 + x_2, x_3) = G(x_1, x_3) + G(x_2, x_3)$$

(the second of the above relations is the Cauchy equation (1), x_3 being a parameter). For arbitrary n the problem of determining the general or general continuous solution of equation (72) is still open.

Equations of the form

(75) $$\varphi(x_1 + x_2, x_3) + \varphi(x_2 + x_3, x_1) + \varphi(x_3 + x_1, x_2) = 0,$$

(76) $$\varphi(x_2, x_1) - \varphi(x_2, x_3) + \varphi(x_3, x_1 + x_2) - \varphi(x_1, x_2 + x_3) = 0,$$

(77) $$\varphi(x_1, x_2) - \varphi(x_2, x_3) + \varphi(x_3, x_1 + x_2) - \varphi(x_1, x_2 + x_3) = 0,$$

as well as related equations, were studied by D. Ž. Đoković [3], [4], D. S. Mitrinović-D. Ž. Đoković [6] and M. Hosszú [14] (cf. also Ghermănescu [3]). The general solution of equation (76) ist given by

$$\varphi(x_1, x_2) = F(x_1 + x_2) - F(x_1) - F(x_2) + 2\,G(x_1) + G(x_2),$$

where $F(x)$ is an arbitrary function and $G(x)$ is an arbitrary solution of the Cauchy equation (1). Equation (75) can also be reduced to the Cauchy equation. The general continuous solution of the more general equation

$$\varphi(x_1 \circ x_2, x_3) + \varphi(x_2 \circ x_3, x_1) + \varphi(x_3 \circ x_1, x_2) = 0,$$

where $x \circ y = g^{-1}[g(x) + g(y)]$ (g, an arbitrary continuous and strictly monotonic function) is an associative operation, is given by

$$\varphi(x_1, x_2) = [g(x_1) + 2\,g(x_2)]\, F(g(x_1) + g(x_2)),$$

where $F(x)$ is an arbitrary continuous function (Đoković [3]).

It is an interesting fact that, although equations (74), (76) and (77) are apparently quite similar, the general solution of (77) is of the form (73) (with an arbitrary F), while the general solutions of (74) and (76) contain solutions of the Cauchy equation (cf. Mitrinović-Đoković [6]).

All the equations discussed in the present section have been linear. An example of a non-linear cyclic equation is provided by

(78) $$\varphi(x_1, x_2)\,\varphi(x_3, x_4) + \varphi(x_1, x_3)\,\varphi(x_4, x_2) + \varphi(x_1, x_4)\,\varphi(x_2, x_3) = 0$$

(equation (78) is cyclic in the variables x_2, x_3, x_4), whose general solution has the form

$$\varphi(x_1, x_2) = \begin{vmatrix} g_1(x_1) & g_1(x_2) \\ g_2(x_1) & g_2(x_2) \end{vmatrix},$$

where g_1, g_2 are arbitrary functions (Mitrinović-Prešić [1], [2]). More general equations whose solution is given by

$$\varphi(x_1, \ldots, x_n) = \begin{vmatrix} g_1(x_1) & \ldots & g_1(x_n) \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ g_n(x_1) & \ldots & g_n(x_n) \end{vmatrix}$$

($g_1, \ldots, g_n$ — arbitrary functions) have been exhibited by L. Carlitz [1] (for $n = 3$) and P. Vasić [1], [2] (arbitrary n). Numerous further generalizations of equation (78) were considered by D. S. Mitrinović, S. B. Prešić and P. Vasić (Mitrinović-Prešić [1], Mitrinović-Prešić-Vasić [1], [2]. Mitrinović-Vasić [1], [2], Vasić [3], [4]).

Equation (78) and its generalizations are related to so called *paracyclic equations* introduced by D. S. Mitrinović [2]. These are equations in which some groups of variables are permuted independently. Let $Q_p(x) = Q_p(x_1, \ldots, x_n) = (x_1, \ldots, x_p)$, $p \leq n$. Equations of the form

$$\sum_{i=1}^{n} \varphi(C_n^{i-1} Q_{p_1}(x_1), C_n^{i-1} Q_{p_2}(x_2), \ldots, C_n^{i-1} Q_{p_k}(x_k)) = 0,$$

where $x_j = (x_{j1}, \ldots, x_{jn})$, $j = 1, \ldots, k$, are called *paracyclic equations of the first kind.* Equations

$$\sum_{i=1}^{n} \{\varphi(C_n^{i-1} Q_{p_1}(x_1), C_n^{i-1} Q_{p_2}(x_2), \ldots, C_n^{i-1} Q_{p_k}(x_k)) +$$
$$+ \varphi(C_n^{i-1} Q_{p_2}(x_1), C_n^{i-1} Q_{p_3}(x_2), \ldots, C_n^{i-1} Q_{p_1}(x_k)) + \cdots +$$
$$+ \varphi(C_n^{i-1} Q_{p_k}(x_1), C_n^{i-1} Q_{p_1}(x_2), \ldots, C_n^{i-1} Q_{p_{k-1}}(x_k))\} = 0$$

are called *paracyclic equations of the second kind.* Paracyclic equations of the first and of the second kind have been considered by D. S. Mitrinović [2] and [6], respectively, under the assumption that the values of φ lie in a module in which condition $\mathcal{A}_m$ is fulfilled for every m. The method developped by him is applicable also to some other functional equations with several unknown functions (cf. Mitrinović [3], [4]).

A number of cyclic functional equations have been discussed also by M. Ghermănescu (cf. Ghermănescu [18], chapter 7). Equations with more involved cyclic operators have been studied by H. Kiesewetter [2] and W. Maier-G. Wutzler [1].

Equations of the form

$$\sum_{i=1}^{n} (-1)^i \varphi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = 0 \tag{79}$$

occur in the theory of homology groups (cf. Hosszú [9]). In the case $n = 3$ (79) becomes the Sinzow equation (62).

14. The equation of invariant curves

Suppose that we are given a transformation on the plane

$$x' = f(x, y), \quad y' = g(x, y) \tag{80}$$

and suppose that a curve

$$y = \varphi(x)$$

is transformed by (80) into itself. Then the function φ must satisfy the functional equation

$$\varphi[f(x, \varphi(x))] = g(x, \varphi(x)), \tag{81}$$

which is therefore called *the equation of invariant curves.*

Equation (81) is the most general equation of type [1, 1, 1], solved with respect to $\varphi[f]$ (cf. in particular formula (6) for $p=1$). It was studied by several authors as e.g. H. Poincaré, J. Hadamard, S. Lattès, and in recent times by P. Montel [2] (where also more details and references may be found) and M. Urabe [3]. All these authors made rather strong regularity suppositions. Recently D. Brydak has investigated continuous solutions of equation (81) under the assumption that the functions f and g are continuous and strictly monotonic with respect to each variable. All these investigations, however, have a local character (the invariant curves are studied in a neighbourhood of a fixed point of transformation (80)).

In the case where the function $f(x, y)$ is invertible with respect to the second variable, equation (81) can be reduced to the simpler form

$$\varphi[\varphi(x)]=g(x, \varphi(x)). \tag{82}$$

Equation (82) has been studied by M. Kuczma [9] and M. K. Fort [2] under the hipotheses that the function $g(x, y)$ is defined, continuous and strictly increasing with respect to each variable in the set

$$\Omega: a \leqslant x \leqslant b, \quad x \leqslant y \leqslant h(x),$$

and moreover $g(a, a)=a$, $g(b, b)=b$, $g(x, x)>x$ for $x \in (a, b)$, $g(x, y)>y$ in int Ω, $g(x, h(x))=h(x)$ for $x \in [a, b]$. Then every continuous solution of (82) must be strictly increasing. The construction of all continuous solutions of (82) is described in Kuczma [9]. Not all these solutions are defined in the whole of $[a, b]$, but there exists a continuous and strictly increasing solution of equation (82) (in general not unique) defined in the whole $[a, b]$ (Fort [2]). It is an open problem what requirements would ensure the uniqueness of a solution of (82).

Equation (82) has also been dealt with by P. E. Lush [1] under quite different conditions (involving differentiability of g).

For the still simpler equation

$$\varphi[\varphi(x)]=g[\varphi(x)]$$

the general solution and the general continuous solution have been given by M. Kuczma [14]. The equation

$$\varphi[\varphi(x)]+\varphi(x)=F(x)$$

has recently been studied by M. Bajraktarević [11].

By a geometrical problem one is led to the functional equation

$$\varphi[x+\varphi(x)]=\varphi(x), \tag{83}$$

which is a particular case of (81). K. Kuratowski [1] has proved that *the only solutions of equation* (83) *with the Darboux property are the functions* $\varphi(x)=const$ (cf. also Wagner [1]). The general continuous solution of the more general equation

$$\varphi[f(x, \varphi(x))]=\varphi(x)$$

has been found by D. Brydak [1] under the assumption that the function $f(x, y)$ is continuous and strictly monotonic with respect to each variable.

Equation (81), as well as most of its particular cases, is fairly difficult. The general solution is not known even for equation (83) (cf. Wagner [1]). It would be very interesting to build a complete theory of continuous solutions of equation (81), which, as for the present, does not exist either.

15. Type [1, 1, 0]

The situation becomes much simpler when the function $f(x, y)$ in (81) does not depend on y. Then we obtain the functional equation

$$\varphi[f(x)] = g(x, \varphi(x)), \tag{84}$$

which has implication index zero.

Equation (84) (which is much easier than (81)) has been extensively studied by many authors and its theory is well developped. M. Kuczma [8], [27] has given its general solution. In the particular case of the equation of automorphic functions (cf. Ghermănescu [18])

$$\varphi[f(x)] = \varphi(x) \tag{85}$$

the general solution has been given also by S. Prešić [2] (cf. also Ghermănescu [18], Kuczma [27]).

J. Kordylewski-M. Kuczma [1] and M. Kuczma [5], [7] have studied equation (84) under the following conditions:

(I) The function $f(x)$ is continuous and strictly increasing in an interval $[a, b]$, $f(a) = a$, $f(b) = b$, $f(x) > x$ in (a, b).

(II) The function $g(x, y)$ is continuous and invertible with respect to y in a region Ω.

(III) Ω_x is a non-degenerated interval and $\Gamma_x = \Omega_{f(x)}$ for $x \in (a, b)$, where $\Omega_x = \{y : (x, y) \in \Omega\}$ is the x-section of the region Ω, and Γ_x is the set of values of the function $g(x, y)$ for $y \in \Omega_x$.

Under hypotheses (I)—(III) *equation* (84) *has an infinity of continuous solutions in the open interval* (a, b). *These solutions may be arbitrarily prescribed on a certain interval.*

The value d assumed by a solution of (84) at $x = b$ must be a root of the equation $d = g(b, d)$. *If the point* (b, d) *belongs to* Ω *and* $g(x, y)$ *fulfils in a neighbourhood of* (b, d) *a Lipschitz condition*

$$|g(x, y_1) - g(x, y_2)| < \vartheta |y_1 - y_2|, \tag{86}$$

where

$$\vartheta < 1, \tag{87}$$

then equation (84) *has an infinity of solutions which are continuous in* $(a, b]$ *and fulfil the condition*

$$\varphi(b) = d. \tag{88}$$

On the other hand, if instead of (86) *and* (87) *the opposite inequalities hold, then equation* (84) *has a unique solution* $\varphi(x)$ *continuous in* $(a, b]$ *and fulfilling* (88). *This solution can be obtained as the limit of a sequence of successive approximations.*

J. Kordylewski [2] has obtained similar results in the case where the function $f(x)$ is decreasing. The continuous dependence of the continuous solutions of equations (84), (90) and (92) on given functions has been proved (under suitable assumptions) by J. Kordylewski and M. Kuczma [4]. The uniqueness of periodic continuous solutions of equation (84) has been proved by M. Kuczma and K. Szymiczek [1].

A number of analogous results concerning equation (84) have been obtained by M. Bajraktarević [4], [5]. Some analogies existing between equations of the form (84) and differential equations have been pointed out by C. Popovici [1], M. Ghermănescu [1], [2] and K. L. Cooke [1].

The linear equation

$$\varphi[f(x)] = g(x)\,\varphi(x) + F(x) \tag{89}$$

has been investigated by J. Kordylewski-M. Kuczma [3], also in the case where $\varphi(x)$ is a complex-valued function of a real variable. The existence of complex-valued solutions of some non-linear equations can be shown under less restrictive hypotheses than (III) (cf. Kuczma-Vopěnka [1]).

Real-valued continuous solutions of the equation

$$\varphi[f(x)] + \varphi(x) = F(x), \tag{90}$$

(which we obtain setting in (89) $g(x) \equiv -1$) have been treated by M. Kuczma [2] and M. Bajraktarević [7] under the assumption that $f(x)$ fulfils (I) and $F(x)$ is continuous in [a, b]. Then, according to the general result concerning equation (84), equation (90) has an infinity of continuous solutions in (a, b), but *a solution continuous in* $(a, b]$ *need not exist.* If it does exist, it is unique and is given by

$$\varphi(x) = \frac{1}{2} F(b) + \sum_{k=0}^{\infty} \{F[f^k(x)] - F(b)\} \tag{91}$$

where the iterates $f^k(x)$ are defined by (57). M. Bajraktarević [7] has proved that the continuous solution (91) of (90) in (a, b] exists e.g. if $F[f(x)] - F(x)$ has a constant sing in a neighbourhood of b (in other words, if $F(x)$ is semimonotonic $\{f\}$ — cf. Kuczma [6]), or if the function $G(x) = F(x) - F(b)$ fulfils the condition

$$0 < \omega\, G(x) < A/n^{\alpha}, \quad G[f(x)]/G(x) < 1 + \frac{1}{n} \quad \text{for } x \in [a_n, a_{n+1}),$$

where $\omega = +1$ or -1, A and α are positive constants and $a_{n+1} = f(a_n)$, $a_0 \in (a, b)$. Similar, but stronger conditions of the existence of the continuous solution in $(a, b]$ of the equation

$$\varphi[f(x)] - \varphi(x) = F(x) \tag{92}$$

(the solution, if it does exist, is unique up to an additive constant and is given by $\varphi(x) = c - \sum_{k=0}^{\infty} F[f^k(x)]$) are to be found in Bielecki-Kisyński [1] and Kordylewski-Kuczma [4] (cf. also McKiernan [1]).

Differentiable solutions of equation (84) have been investigated by B. Choczewski [2], [3]. *If hypotheses* (I)—(III) *are fulfilled and moreover the functions* $f(x)$ *and* $g(x, y)$ *are of class* C^r $(1 \leqslant r \leqslant \infty)$, $f'(x) > 0$ *in* (a, b), *then equation* (84) *has an infinity of solutions of class* C^r *in* (a, b). *These solutions may be prescribed arbitrarily on a certain interval*[11]. *If moreover*

$$\frac{1}{[f'(b)]^r} \left| \frac{\partial g}{\partial y}(b, d) \right| < 1 \tag{93}$$

[11] For the Abel equation (100) this theorem had previously been proved by U. T. Bödewadt [1].

and r-th derivatives of the function $g(x, y)$ *satisfy some further conditions, then equation* (84) *has an infinity of solutions which are of class* C^r *in* $(a, b]$ *and fulfil suitable initial conditions*

$$\varphi(b)=d, \quad \varphi'(b)=d_1, \ldots, \quad \varphi^{(r)}(b)=d_r. \tag{94}$$

On the other hand, if instead of (93) *the converse inequality is fulfilled, then equation* (84) *has a unique solution which is of class* C^r *in* $(a, b]$ *and fulfils conditions* (94).

For the linear equation (89) (where φ may assume values in a Banach space) a corresponding result has been obtained by M. Kuczma [13].

Let us notice, however, that we know very little about the „indeterminate case" where

$$\left|\frac{\partial g}{\partial y}(b, d)\right|=[f'(b)]^r, \qquad r \geqslant 0. \tag{95}$$

In the case of equation (89) relation (95) with $r=0$ means that $|g(b)|=1$. In such a case only equations (90) and (92) were studied more in detail (cf. however the recent paper by B. Choczewski and M. Kuczma [1]). For $r=1$ the corresponding case was treated for the Schröder equation (99) (cf. Kuczma [20]). But a great variety of problems are still left open.

In the complex domain equation (84) has been dealt with by A. H. Read [1]. He proved that *if the function* $f(x)$ (*complex-valued of a complex variable*) *is regular in a neighbourhood of a point b such that* $f(b)=b$, $0<|f'(b)|<1$, *and if the function* $g(x, y)$ (*complex-valued of two complex variables*) *is regular in a neighbourhood of a point* (b, d), *where d fulfils* $d=g(b, d)$, *and if*

$$\left|\frac{\partial g}{\partial y}(b, d)\right|>|f'(b)|,$$

then equation (84) *has a unique solution* $\varphi(x)$ *regular in a neighbourhood of b and fulfilling* (88).

Sen-ichiro Tanaka [1] was concerned with the case $f(x)=x+1$ of (84), also in the complex domain. Equations (85) and (89) on the complex plane have been studied by P. J. Myrberg [4], [5].

The homogeneous linear equation

$$\varphi[f(x)]=g(x)\,\varphi(x) \tag{96}$$

was treated by V. Ganapathy Iyer [1], [2], [4], [5], esentially under the condition that one of the occurring functions is a polynomial. V. Ganapathy Iyer was concerned with entire solutions of equation (96) on the complex plane. The particular cases $f(x)=x+a$, $f(x)=ax$ and $f(x)=x^2$ are treated in detail. Especially the equation

$$\varphi(ax)=g(x)\,\varphi(x)$$

has attracted the attention of several authors (Ganapathy Iyer [2], Schweizer [1], Vaida [1], Vâlcovici-Vaida [1]).

Systems of equations of form (84)

$$\varphi_i[f(x)]=g_i(x, \varphi_1(x), \ldots, \varphi_n(x)), \qquad i=1, \ldots, n, \tag{97}$$

have been studied to a much lesser extent (cf. Bajraktarević [5], Majcher [1]). Such a system can be reduced to a single equation of order n. In the linear case system (97) can be written as the matrix equation

$$\Phi[f(x)] = G(x)\,\Phi(x) + F(x), \tag{98}$$

where G is an $n \times n$ matrix and Φ and F are $n \times 1$ matrices. Equation (98) (where x is a point in an m-dimensional space) have been considered by M. Ghermănescu [13] from a rather general point of view (without discussing the regularity of solutions).

16. The Schröder equation and related equations

There are some particular cases of equation (84) that are of a great importance and have been studied very extensively. Here we mention the Schröder equation

$$\varphi[f(x)] = s\,\varphi(x), \tag{99}$$

the Abel equation

$$\varphi[f(x)] = \varphi(x) + 1, \tag{100}$$

the Böttcher equation

$$\varphi[f(x)] = [\varphi(x)]^m, \tag{101}$$

and the Poincaré equation

$$\varphi(sx) = F[\varphi(x)] \tag{102}$$

(cf. Schröder [1], Abel [1], Picard [1]). These equations were thoroughly investigated in the years 1919—1924 by P. Fatou, G. Julia and others (some references may be found in Kuczma [20]).

The Schröder equation is perhaps the most important one. The remaining equations can be reduced (under suitable assumptions) to (99): equation (100) by putting $\psi(x) = s^{\varphi(x)}$, equation (101) by putting $\psi(x) = \log \varphi(x)$, equation (102) by putting $\psi(x) = \varphi^{-1}(x)$. These equations are mainly connected with the iteration theory, but they arise also in numerous other questions, like the investigations of the invariants of the local transformations of the real line (equation (99); cf. Sternberg [1], Kuczma [20]), or the investigations of the distribution of zeros of solutions of some differential equations (equation (100); cf. Barvinek [1]).

G. Koenigs [1] (cf. also Kneser [1], Jabotinsky [1]) proved that *if the function $f(x)$ (complex-valued of a complex variable) is analytic in a neighbourhood of* 0, $f(0)=0$, $f'(0)=s$, $0<|s|<1$, *then equation* (99) *has a unique solution $\varphi(x)$ which is analytic in a neighbourhood of* 0 *and such that* $\varphi'(0)=1$. *This solution is given by*

$$\varphi(x) = \lim_{n\to\infty} s^{-n} f^n(x), \tag{103}$$

where $f^n(x)$ are iterates of $f(x)$ (cf. (57)).

H. Kneser [1] showed that instead of analyticity of $f(x)$ it is enough to assume that

$$f(x) = sx + O(|x|^{1+\delta}), \quad 0<s<1, \quad \delta>0, \quad x\to 0,$$

(where $f(x)$ may be a real-valued function of a real variable as well). Then of course, function (103) need not be analytic, not even of class C^1 or strictly monotonic in a neighbourhood of 0, although the derivative $\varphi'(0)$ still exists and equals 1. Since the existence of φ^{-1} is very important in many applications, the following theorem due to G. Szekeres [1] is more useful.

It the function $f(x)$ is of class C^1 and strictly increasing in an interval $[0, a)$, $0<f(x)<x$ in $(0, a)$ and if

$$f'(x) = s + O(x^{\delta}), \quad \delta > 0, \quad 0 < s < 1, \quad x \to 0, \tag{104}$$

then the Schröder equation has a unique solution $\varphi(x)$ which is of class C^1 and strictly increasing in $[0, a)$ and such that $\varphi'(0) = 1$. This solution is given by formula (103).

Condition (104) is not superfluous. If it is not fulfilled, then (99) can happen to have an infinity of solutions of class C^1 in $[0, a)$ (these solutions may be prescribed arbitrarily on a certain interval), or none except for the trivial one $\varphi(x) \equiv 0$. For a complete discussion of C^1 solutions of the Schröder equation cf. Kuczma [20].

G. Szekeres [1] has proved also a number of results concerning real solutions of the Schröder equation in the case where $f'(0) = 1$ or $f'(0) = 0$, and the complex solutions of the Schröder equation in an angular domain. M. Kuczma [20] (cf. also Kuczma [19], [23]) has proved that *if the function $f(x)$ is convex or concave and strictly increasing in* $(0, a)$, $0<f(x)<x$ in $(0, a)$, $f(0) = 0$, $f'(0) = s$[12], $0<s<1$, *then equation* (99) *has a unique one-parameter family of convex/concave solutions in* $(0, a)$. *These solutions are given by*

$$\varphi(x) = c \lim_{n \to \infty} f^n(x)/f^n(d), \tag{105}$$

where d is an arbitrarily chosen point from $(0, a)$ *and c is an arbitrary constant.*

Limit (105) is more general than (103), but is identical with (103) whenever the latter exists. Functions (105), if they exist, are called the principal solutions of the Schröder equation (Szekeres [1], Kuczma [20]). They are characterized among all the continuous, strictly increasing solutions of (99) (such solutions may be prescribed arbitrarily on a certain interval; cf. Walker [1], Szekeres [1], Kuczma [20]) by the best behaviour near zero. The principal solutions of the Schröder equation are unique up to a multiplicative constant.

The Schröder equation for functions of several variables and various generalizations have been treated among others by N. Pastidès [1], P. Montel [2], M. Kuczma [20], M. Urabe [1]—[4].

The Abel equation (100) is perhaps less general than (99). R. Tamb Lyche [1] proved that (100) *has a solution in a set E if and only if*

$$f^k(x) \neq x \quad \text{for} \quad k = 1, 2, 3, \ldots, \text{ and } x \in E. \tag{106}$$

If (106) is fulfilled in an interval E, then the Abel equation has an infinity of solutions in E (they may be prescribed arbitrarily on a certain interval) having the same regularity as $f(x)$ (Bödewadt [1]; cf. the preceding section). Therefore in order to obtain a uniqueness of solutions one must, instead of assuming analytic conditions at a fixed point of $f(x)$, make some assumptions

[12] $f'(0)$ denotes the right-sided derivative of $f(x)$ at 0, which necessarily exists, since $f(x)$ is convex or concave.

concerning the asymptotic behaviour of $\varphi(x)$ (cf. Szekeres [1], [2], [3]). So G. Szekeres [3] has proved that *there is a unique function* $\varphi_0(x)$, *normalized to fulfil* $\varphi_0(1)=0$, *which satisfies the equation*

$$\varphi_0(e^x-1)=\varphi_0(x)+1, \qquad x>0, \tag{107}$$

and the condition

$$(-1)^{k+1}\varphi_0^{(k)}(x)>0 \quad \text{for} \quad k=1, 2, \ldots, \text{ and } x>0 \tag{108}$$

(in other words, $\varphi_0(x)$ is totally monotonic; here $\varphi_0^{(k)}(x)$ denotes the k-th derivative of $\varphi_0(x)$). He also showed that *for every L-function* $f(x)$ *there exists a solution* $\varphi(x)$ *of equation* (100), *unique up to an additive constant, such that*

$$\frac{\varphi'(x)}{\varphi_0'(x)} \to \text{const as } x\to\infty. \tag{109}$$

Here L-functions are Hardy's logarithmico-exponential functions, i.e. members of the smallest set $\boldsymbol{H}$ which contains the constant functions $f(x)=\text{const}$, the identity function $f(x)=x$ and is closed under the rational operations and the application of exp ( ) and log $|\;|$.

As we see, the function $\varphi_0(x)$ plays here an exceptional rôle. G. Szekeres proposes to accept $\varphi_0(x)$ as a new standard order of infinity. Seven place tables of $\varphi_0(x)$ and $\varphi_0'(x)$ have been supplied by K. W. Morris-G. Szekeres [1].

Equations (101) and (102) have a more narrow field of applications and therefore they have attracted less attention. A generalization of equation (102) has been dealt with by N. Pastidès [2].

17. Iterations

A one parameter family of functions $f^u(x)$, defined in a neighbourhood of $x=0$, is called *an iteration group of the function* $f(x)=f^1(x)$ provided that

$$f^u[f^v(x)]=f^{u+v}(x) \tag{110}$$

holds for every pair of $u, v\in(-\infty, +\infty)$ in a suitable neighbourhood of 0. (110) is the equation of translation (cf. § 10). Consequently (compare formula (56))

$$f^u(x)=\varphi^{-1}[\varphi(x)+u], \tag{111}$$

where $\varphi(x)$ is an invertible solution of the Abel equation (100) (cf. among others Ward-Fuller [1], Bajraktarević [3], Bödewadt [1], Michel [1]). Alternatively $f^u(x)$ may be defined as

$$f^u(x)=\varphi^{-1}(s^u\varphi(x)), \tag{112}$$

where $\varphi(x)$ is an invertible solution of the Schröder equation (99).

Since invertible solutions of equations (99) or (100) are not unique, an iteration group of a function is not unique either. A detailed discussion of this question in the real case is to be found in Michel [1].

If $f'(0)=s$, $0<s<1$, then an iteration group $f^u(x)$ of $f(x)$ is called *regular* whenever

$$\lim_{x\to 0}\frac{f^u(x)}{x}=s^u \qquad \text{for every } u.$$

The regular iteration group, if it exists, is unique. In fact, if (112) defines the regular iteration group of $f(x)$, then $\varphi(x)$ must be the principal solution of the Schröder equation. The converse is not true, for the principal solution of equation (99) need not be invertible. But if $\varphi(x)$ is an invertible principal solution of equation (99), then the iteration group defined by (112) is regular (Szekeres [1], Kuczma [20], Lundberg [1]). Therefore condition (104), or a convexity or concavity of a function $f(x)$ (fulfilling further conditions as in § 16) is a sufficient condition for the existence of the regular iteration group of $f(x)$ (Szekeres [1], Kuczma [20], Lundberg [1], Fort [1]).

Similar notions can be defined also if $f'(0)=0$ or $f'(0)=1$ (Szekeres [1], [2], [3], Michel [1], Lundberg [1]). Sometimes it is convenient to consider iteration groups of a function in a neighbourhood of the infinity; then relation (110) is postulated for x sufficiently large. If $f(x)=\varphi_0^{-1}[g(\varphi_0(x))]$ (where φ_0 is the solution of equation (107) fulfilling (108)) and $g(x)=x+w(x)$, $\lim\limits_{x\to\infty}\frac{w(x)}{x}=0$, then the iteration group $f^u(x)=\varphi_0^{-1}[g^u(\varphi_0(x))]$ of $f(x)$, where $g^u(x)=x+w^u(x)$ is an iteration group of $g(x)$, is called *regular* whenever $\lim\limits_{x\to\infty}\frac{w^u(x)}{w(x)}=u$. Now, *if equation* (100) *has a solution* $\varphi(x)$ *fulfilling* (109),[13] *then* $f(x)$ *has a unique regular iteration group, which is given by* (111). In particular, *every L-function possesses a unique regular iteration group* (Szekeres [3]).

If $f(x)$ is analytic in a neighbourhood of the origin

$$f(x)=a_1x+a_2x^2+\cdots,$$

then the coefficients of the expansion of $f^u(x)$

$$f^u(x)=b_{u1}x+b_{u2}x^2+\cdots \tag{113}$$

may be easily calculated. This goes back as far as J. G. Tralles 1814 and C. G. J. Jacobi 1825. The convergence of series (113), however, is a very delicate matter. If $0<|a_1|<1$, then series (113) converge and provide an iteration group of $f(x)$ analytic in x and u (Jabotinsky [1]). The case $a_1=1$ has been investigated by G. Szekeres [1], P. Erdös-E. Jabotinsky [1], B. Muckenhoupt [1] and I. N. Baker [5]. The latter has proved that the set of u's for which series (113) has a positive radius of convergence may consist of the whole complex plane, or of a one or two dimensional lattice of points. In particular, for $f(x)=e^x-1$ series (113) has a positive radius of convergence if and only if u is an integer (Baker [2]).

Iterates of complex orders u have been treated by H. Töpfer [1].

M. A. McKiernan [2] proved that the curves given on the complex plane by $z=f^u(z_0)$, $u\in(-\infty,+\infty)$, yield the solution of a variational problem.

Closely related to the iteration theory is that of commutable functions. Functions $f(x)$ and $\varphi(x)$ are called *commutable* (or *permutable*) if

$$\varphi[f(x)]=f[\varphi(x)]. \tag{114}$$

If $f(x)$ is given, (114) is a particular case of equation (84).

Commutable functions and related classes of functions have been studied by many authors (Baker [2], [4], [5], Berg [2], Block-Thielman [1], Fort [1], Ganapathy Iyer [3], Hadamard [1], Hällström [1], [2], [3], Hosszú [10], [12],

[13] Such a solution is unique up to an additive constant.

Jacobsthal [1], Kuczma [18], [20], Nikolaus [1], Ritt [1]). In particular there are some theorems to the effect that, under suitable conditions, the only functions commuting with a given $f(x)$ are the regular iterates of $f(x)$ (Hadamard [1], Berg [2], Fort [1], Kuczma [18], [20]).

We end this section with a mention of the following problem (which, as far as we know, has been raised by J. R. Isbell). Suppose that $f_1(x)$ and $f_2(x)$ are commuting continuous mappings of [0, 1] onto itself. Do they have a common fixed point (i.e. does there exist an $x_0 \in [0, 1]$, such that $f_1(x_0) = f_2(x_0) = x_0$)? In spite of its apparent simplicity, this problem turns out very difficult and remains still unsolved except in certain special cases.

18. Further particular cases of type [1, 1, 0]

The problem of Goursat for the differential equation

$$\frac{\partial^2 z}{\partial x\, \partial y} = G(x, y) \tag{115}$$

leads to functional equation (92) (Bielecki-Kisyński [1]). Theorems concerning the existence and uniqueness of a solution of the problem of Goursat for equation (115) follow directly from the corresponding theorems for equation (92). The problem of Goursat for more complicated differential equations than (115) leads to some functional equations of higher orders (Majcher [2]).

The equation

$$\varphi(2x) = \frac{1}{2}[\varphi(x) + x]$$

finds an application in the statics (Pompeiu [1]). N. Gersevanov [1] showed that some particular examples of equation (84) can be used in the hydromechanics. The equation.

$$\varphi(x^2) + \varphi(x) = x$$

has been used by H. Steinhaus [1] in investigations of the convergence of a certain power series.

P. J. Myrberg [3] has studied the non-linear equation

$$\varphi(kx) = [\varphi(x)]^2 + p$$

in the complex domain. The equation

$$\varphi(x^2) + [\varphi(x)]^2 + 2x = 0$$

has occurred in connection with a combinatorial problem in non-associative algebras (Etherington [1], [2]). The similar equation

$$\varphi(x^2) - [\varphi(x)]^2 = h(x)$$

has been treated by I. N. Baker [3] and J. Lambek-L. Moser [1] in connection with a problem in the number theory.

Also a number of further equations occur in the number theory (e.g. Berg [1], Maier-Krätzel [1]). Perhaps the most important one is the Riemann equation

$$\pi^{-\frac{x}{2}} \Gamma\left(\frac{x}{2}\right) \varphi(x) = \pi^{-\frac{1-x}{2}} \Gamma\left(\frac{1-x}{2}\right) \varphi(1-x) \tag{116}$$

and its various generalizations. Of the modern work in this connection we mention here Apostol-Sklar [1], Bochner [1], Bochner-Chandrasekharan [1], Chandrasekharan-Mandelbrojt [1], Chandrasekharan-Narasimhan [1], [2], [3], Kahane-Mandelbrojt [1], Mandelbrojt [1], [2], Siegel [1].

M. Ghermănescu [12] considered functions satisfying simultaneously two equations

$$\varphi[f_1(x)] = g_1(x)\,\varphi(x), \quad \varphi[f_2(x)] = g_2(x)\,\varphi(x)$$

as a generalization of doubly periodic functions. W. Sierpiński [1], has proved that the Lebesgue's singular function may be characterized as the only continuous solution of the simultaneous functional equations

$$\varphi\left(\frac{x}{3}\right) = \frac{1}{2}\,\varphi(x), \quad \varphi\left(\frac{x+1}{3}\right) = \frac{1}{2}, \quad \varphi\left(\frac{x+2}{3}\right) = \frac{1}{2} + \frac{1}{2}\,\varphi(x), \qquad x \in [0, 1].$$

M. Kuczma [19] has characterized the exponential and logarithmic functions as the differentiable or convex solutions of the equations

$$\varphi(2x) = [\varphi(x)]^2 \quad \text{and} \quad \varphi(x^2) = 2\,\varphi(x),$$

respectively. G. de Rham [1], [2], [3] has shown that many continuous and nowhere differentiable functions can be elegantly obtained as solutions of some particular functional equations of form (84).

J. G. Mikusiński [1] has observed that $\varphi(x) = \cos x$ is the only analytic solution of the equation

$$\varphi(2x) = 2\,[\varphi(x)]^2 - 1, \tag{117}$$

fulfilling the condition $\varphi(0) = 1$, $\varphi''(0) = -1$. H. G. Forder [1] has proved that the functions $\cos kx$ and $\cosh kx$ are the only real solutions of (117) which are non-constant, continuous, even and twice differentiable at $x = 0$. The latter condition cannot be weakened. If we assume only that $\varphi(x)$ is once differentiable at zero, then even the additional hypothesis of convexity does not guarantee that $\varphi(x) = \cosh kx$ (Cooper [1]). The function $\varphi(x) = \cos x$ may be characterized, however, as the only continuous solution of equation (117) which is periodic with period 2π and fulfils the condition

$$\varphi(x) > 0 \quad \text{for} \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \varphi(x) < 0 \quad \text{for} \quad x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right),$$

(Kuczma [22]). (The latter condition is rather disagreeable and it would be desirable to find weaker conditions that would characterize the cosine among the solutions of equation (117)). Similarly L. Dubikajtis [1] has proved that $\varphi(x) = \sin x$ is the only continuous solution of the equation

$$\varphi\left(\frac{\pi}{2} - 2x\right) + 2\,[\varphi(x)]^2 = 1$$

which is odd, periodic with period 2π and positive in $\left(0, \frac{\pi}{2}\right)$. The uniqueness of continuous periodic solutions of the more general equation (84) has been established (under suitable conditions) by M. Kuczma and K. Szymiczek [1].

The existence and uniqueness of periodic solutions of the equation

$$\varphi(x) - \varphi(2x) = F(x)$$

has been proved by Z. Ciesielski [1]. Periodic and almost periodic solutions of difference equations have been investigated by A. Halanay [1]. Periodic solutions of some equations of form (92) are of a great importance in the study of some classical problems of Poincaré and Denjoy (the restricted three body problem, mappings of a circle or of an annulus onto itself; cf. Arnold [6], Moser [1]).

19. Monotonic and convex solutions

Euler's function $\Gamma(x)$ satisfies the functional equation

(118) $$\Gamma(x+1)=x\,\Gamma(x), \qquad x>0,$$

and the condition

(119) $$\Gamma(1)=1.$$

But equation (118) possesses also other solutions fulfilling condition (119). Therefore, in order to characterize Euler's function with the aid of relations (118) and (119) one must set further requirements.

In the year 1931 E. Artin [1] proved that *Euler's function is the only logarithmically convex*[14] *function satisfying equation* (118) *and condition* (119). Alternatively, we can characterize Euler's function as the only solution of (118) and (119) which is asymptotically equal to $\left(\frac{x}{e}\right)^x \left(\frac{2\pi}{x}\right)^{1/2}$ as well as by similar conditions (Anastassiadis [3], [6]). Also in the complex domain the function $\Gamma(x)$ can be defined by (118), (119) and some additional conditions (Nörlund [1], Picard [1], Schmidt [1]).

Artin's theorem has been generalized by W. Krull [1] (cf. also Kuczma [1], Dinghas [1]) as follows.

If the function $F(x)$ is concave in $(0, \infty)$ and fulfils the condition $\lim_{k\to\infty}\{F(k+1)-F(k)\}=0$, then there exists exactly one convex function $\varphi(x)$ satisfying for $x>0$ the equation

(120) $$\varphi(x+1)-\varphi(x)=F(x)$$

and the condition

(121) $$\varphi(1)=y_0.$$

This function is given by

$$\varphi(x)=y_0-F(x)+x\,F(1)-\sum_{k=1}^{\infty}\{F(k+x)-F(k)+x\,[F(k+1)-F(k)]\}.$$

A further generalization has been given by M. Kuczma [21] (cf. also Krull [1]), who proved that *if the function $F(x)$ is concave of order $n\geqslant 0$ and fulfils the condition*[15] $\lim_{x\to\infty}\Delta_1^n F(x)=0$, *then there exists exactly one function*

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[14] I.e. its logarithm is a convex function.

[15] The operator Δ_h^k is defined by relation (28). A function $f(x)$ is called concave of order n if its divided difference (cf. Nörlund [1]), of order n is non-positive or, what ammounts to the same, if $f(x)$ is measurable and $\Delta_h^{n+1} f(x)\leqslant 0$ in the interval considered (and analogically for functions convex of order n).

$\varphi(x)$ *which is convex of order* n *and satisfies equation* (120) *and condition* (121). (The formula for this solution is rather complicated). As a consequence he derived the following characterization of polynomials: *A function* $\varphi(x)$, *convex of order* n, *satisfies the equation* $\Delta_1^{n+1}\varphi(x)=0$ *if and only if* $\varphi(x)$ *is a polynomial of degree* $\leqslant n$. In other words, *the polynomials of degree* $\leqslant n$ *are characterized among measurable functions by the relations*

$$\Delta_1^{n+1}\varphi(x)=0, \quad \Delta_h^{n+1}\varphi(x)\geqslant 0, \quad x\in(-\infty,+\infty), \quad h\in(0,+\infty).$$

This improves a result of Th. Angheluţa (cf. § 6, in particular relation (27)).

For equation (92), which is a generalization of (120), a theorem on the uniqueness of convex solutions has been proved by M. Kuczma [3]. He also proved that *if the function* $f(x)$ *fulfils hypothesis* (I)[16] *and the function* $F(x)$ *is monotonic in* (a, b) *and* $\lim_{x\to b} F(x)=0$, *then there exists exactly one monotonic function* $\varphi(x)$ *satisfying equation* (92) *and the condition* $\varphi(x_0)=y_0$, $x_0\in(a, b)$. *This function is given by*

$$\varphi(x)=y_0-\sum_{k=1}^{\infty}\{F[f^k(x)]-F[f^k(x_0)]\},$$

where the iterates $f^k(x)$ *are defined by* (57) (Kuczma [12], [10], [20]). Some theorems concerning monotonic solutions have been proved for equation (90) by M. Kuczma [6], [7], for the more general equation

$$\varphi[f(x)]-p\,\varphi(x)=F(x)$$

(p—a constant) by D. Brydak-J. Kordylewski [1], and for the equation in general form (84) by M. Bajraktarević [4].

Some functions related to $\Gamma(x)$ also can be characterized by functional equations and conditions of convexity or monotonicity (Anastassiadis [7]). So e.g. A. Mayer [1] has proved that *the function*

$$\Phi(x)=\frac{1}{\sqrt{2}}\cdot\Gamma\left(\frac{x}{2}\right)\Big/\Gamma\left(\frac{x+1}{2}\right)$$

is the only convex solution of the equation

$$\Phi(x+1)=\frac{1}{x\,\Phi(x)}.$$

J. Anastassiadis [1] has proved that in the above theorem the condition of convexity may be replaced by weaker conditions (semi-convexity or semi-monotonicity). He also proved (Anastassiades [2], [4]) that *the only logarithmically convex or monotonic solution of the equation*

$$\Phi(x+1)=\frac{1}{x+y}\,\Phi(x), \qquad x>0,$$

(y—*a fixed parameter*) *fulfilling the condition* $\Phi(1)=\dfrac{1}{y}$ *is the function*

$$\Phi(x)=\mathrm{B}(x, y)=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

[16] This refers to § 15. Here a and b may be infinite as well.

Theorems of Mayer and Anastassiadis have been generalized by M. Bajraktarević [1], M. Kuczma [6], [7], [10] and by J. Anastassiadis himself (Anastassiadis [5]).

Let us notice that the function $\Gamma(x)$ satisfies also the equation

$$\Gamma(x)\,\Gamma(1-x)=\frac{x}{\sin\pi x}, \tag{122}$$

but it is not so useful as (118) for a characterization of this function.

20. Equations of higher orders

The theory of functional equations of higher orders (of rank 1 and implication index zero) is not so well developped as that of equation (84). Such equations have the form

$$F(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)])=0. \tag{123}$$

A way to obtain the general solution of equation (123) has been indicated by S. B. Prešić [3].

Some theorems concerning equation (123), analogous to theorems on continuous solutions of equation (84), have been proved by M. Bajraktarević [5], B. Choczewski [1], J. Kordylewski [1], J. Kordylewski-M. Kuczma [2], G. Majcher [1]. In particular B. Choczewski [1] considered the equations

$$\varphi(x)=H(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]), \tag{124}$$

$$\varphi[f_n(x)]=G(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_{n-1}(x)]), \tag{125}$$

where the functions $f_i(x)$ fulfil conditions (I) of § 15 and moreover $f_1(x)\leqslant f_j(x)\leqslant f_{n-1}(x)<f_n(x)$ for $j=2, \ldots, n-2$, and the functions G and H are continuous in

$$\Omega=[a, b]\times\{(\alpha, \beta)\}^n,$$

$\alpha<H<\beta$, $\alpha<G<\beta$ in Ω. Let d be a number fulfilling $H(b, d, \ldots, d)=d$, $G(b, d, \ldots, d)=d$, $\alpha<d<\beta$. *If there exist numbers* $a_i\geqslant 0$ *such that* $0<\sum_{i=0}^{n} a_i<1$ *and*

$$|H(x, y_1, \ldots, y_n)-H(x, y_1', \ldots, y_n')|\leqslant\sum_{i=1}^{n} a_i\,|y_i-y_i'| \tag{126}$$

holds in a neighbourhood of (b, d), *then equation* (124) *has a unique solution* $\varphi(x)$ *continuous in* $(a, b]$ *and fulfilling* (88). *If, on the other hand,* (126) *holds with G instead of H, then equation* (125) *has an infinity of solutions continuous in* $(a, b]$ *and fulfilling* (88) (*these solutions may be arbitrarily prescribed on a certain interval*). But we know very little about the behaviour of continuous solutions of equation (123) in the case where none of the corresponding functions H and G fulfils condition (126).

The particular case where $f_k(x)=f^k(x)$ are iterates of a function $f(x)$ and the function $F(x, y_0, \ldots, y_n)$ is linear with respect $y_0, \ldots, y_n$, has more often been dealt with. Equation (123) has then the form

$$\varphi[f^n(x)]+A_1(x)\,\varphi[f^{n-1}(x)]+\cdots+A_n(x)\,\varphi(x)=B(x). \tag{127}$$

Equations of form (127) have been studied by C. Popovici [1], M. Ghermănescu [5], [8], [9], [11], [13], [15], [21] (cf. also Ghermănescu [18]) J. Kordylewski-M. Kuczma [3], M. Bajraktarević [8], G. Majcher [1], D. S. Mitrinović [8]. In particular, M. Ghermănescu [5] has proved that *the general solution of equation* (127) *depends on at most n arbitrary automorphic functions*[17]. In the case where the coefficients A_i are constant and the function $f(x)$ fulfils hypothesis (I) of § 15, the theory of continuous solutions of equation (127) may be regarded as complete (Kordylewski-Kuczma [3], Bajraktarević [8]). Differentiable solutions of equation (127) have been dealt with by G. Majcher [1]. The important case where the function $f(x)$ fulfils the relation

$$f^{n+1}(x) \equiv x \tag{128}$$

(for $n=1$ (128) is fulfilled e.g. in the case of equations (116) and (122)) has been investigated by M. Ghermănescu [8], [11] and D. S. Mitrinović [8] (cf. also Aczél-Ghermănescu-Hosszú [1]). A somewhat more general linear equation has recently been considered by S. B. Prešić [4], [5].

Integral equations with two variable limits of integration

$$\psi(x) - \lambda \int_{f(x)}^{x} K(x, s)\, \psi(s)\, ds = 0$$

can be reduced to a system of integral equations with constant limits of integration and of functional equations of form (127) (Ghermănescu [16], [19]).

E. Vincze [7] has proved that the only complex-valued, non-constant solution of the equation

$$\varphi(x) = \varphi(ax)\, \varphi(bx), \quad a > 0, \quad b > 0, \quad a^2 + b^2 = 1, \tag{129}$$

twice differentiable at $x=0$ is $\varphi(x) = e^{mx^2}$. Equation (129) occurs in the probability theory.

The regular solution of the equation[18]

$$\varphi(4x, 4y) - 4\varphi(x, y) = a\{\varphi(2x, -2y)\}^2 + b\{\varphi(2x, -2y)\}^5$$

yields an example of an entire function $\varphi(x)$ (whose values as well as arguments are couples of complex numbers) which maps homeomorphically the whole space of two complex variables onto its proper part (Bochner-Martin [1]).

Further particular cases of equation (123), where φ is a more-place function, have been solved by M. Ghermănescu [18], D. S. Mitrinović-D. Ž. Đoković [12], D. S. Mitrinović-P. M. Vasić [3], M. Kuczma [28].

21. Finite differences

A large class of equations of form (123) has gained a particular popularity. We are speaking here about difference equations. These are equations of form (123) where $f_k(x) = x + kh$, where $h > 0$ is a constant. Difference equations find many applications in problems of approximate solutions of partial differential equations (cf. e.g. Ladyženskaya [1]). They find also numerous

[17] These are functions satisfying equation (85).

[18] This equation can be written as

$$\varphi[f^2(p)] - a\{\varphi[f(p)]\}^2 - b\{\varphi[f(p)]\}^5 - 4\varphi(p) = 0,$$

where $p = (x, y)$ and $f(p) = (2x, -2y)$, so it is an equation of form (123).

applications directly in physics (e.g. Derfler [1], Okabe [1], Williams [1]; cf. also Ghermănescu [7], Halanay [1]), as well as in other branches of science (Lush [1], Goldberg [1]). The theory of difference equations is well developped nowadays and has several times been exposed in books. It would be impossible to present it here, even superficially. The interested readers are referred to the classical work of Nörlund [1] or to modern monographs like Goldberg [1] or Levy-Lessman [1].

We would like, however, to mention here a particular class of difference equations, viz. recurrent formulae (cf. Montel [2]). Theorems concerning the convergence of a sequence defined by the recurrent formula

$$a_{\nu+n}=G(a_\nu,\ a_{\nu+1},\dots,\ a_{\nu+n-1}) \tag{130}$$

can be derived from the limit properties of solutions of the functional equation

$$\varphi(x+n)=G(\varphi(x),\ \varphi(x+1),\dots,\ \varphi(x+n-1)).$$

In the case where the function G is linear a detailed discussion of the dependence of the convergence of a sequence a_ν defined by relation (130) upon a choice of the initial terms $a_0,\dots,a_{n-1}$ has been given by M. Kucharzewski-M. Kuczma [2]. For arbitrary functions G theorems guaranteeing the convergence of the sequence a_ν independently of a choice of the initial terms have been proved by J. Aczél [2], [6] and A. G. Azpeitia [1], [2]. The latter proved the following, fairly general theorem.

If a sequence of (complex-valued) functions $G_k(z_0,\ z_1,\dots,z_{n-1})$ converges uniformly to $G(z_0,\ z_1,\dots,\ z_{n-1})$ for $z_0,\ z_1,\dots,\ z_{n-1}$ belonging to a convex domain D on the complex plane, and if for every $k\geqslant 0$ the point $w_k=G_k(z_0,\ z_1,\dots,\ z_{n-1})$ always belongs to the convex hull R_k of the points $z_0,\ z_1,\dots,\ z_{n-1}$ and is different from the extreme points of R_k (except when $z_0=z_1=\dots=z_{n-1}$; then also $w_k=z_0$), then the sequence a_ν defined by

$$a_{\nu+n}=G_\nu(a_\nu,\ a_{\nu+1},\ \dots,\ a_{\nu+n-1})$$

converges independently of how the initial terms $a_0,\ \dots,\ a_{n-1}$ have been chosen in D.

It is an interesting fact that from properties of recurrent sequences one can obtain some theorems on roots of algebraic equations (Aczél [2], Kucharzewski-Kuczma [2]).

The existence and uniqueness of monotonic sequences generated by some simple recurrence formulae have been investigated by M. Kuczma [16] and D. Brydak-J. Kordylewski [1].

Closely connected with difference equations are *q-difference equations* or *geometrical difference equations*. These are equations of form (123) where $f_k(x)=q^k x$, q being a constant, $0<|q|<1$. Analytical theory of linear q-difference equations has been built by W. J. Trjitzinsky [1]. In modern times q-difference equations have been extensively studied by W. Hahn (cf. e.g. Hahn [1]—[5]).

22. Iterated equations

Only a few types of functional equations with a positive implication index have been wider studied. Perhaps the most important one is the Babbage equation (Babbage [1])

$$\varphi^n(x)=x \tag{131}$$

(φ^n denotes the n-th iterate of φ), which has been treated by many authors. E. Vincze [1] has proved that *if n is odd, then the only monotonic solution of* (131) *is $\varphi(x)=x$; if n is even, then every monotonic solution of* (131) *must be an involutory function* (i.e. it must fulfil $\varphi^2(x)=x$). On the other hand, *every continuous solution of equation* (131) *must be strictly monotonic.*

The equation

$$\varphi^n(x)=g(x) \tag{132}$$

is an immediate generalization of the Babbage equation. The general solution of equation (132) in a set E such that $g(E)=E$ has been constructed by S. Łojasiewicz [1] (cf. also Isaacs [1], Haïdukov [1]). Continuous solutions of (132) have been investigated by M. Kuczma [15] and P. I. Haïdukov [1] under the assumption that the function $g(x)$ is continuous and strictly monotonic. It would be very desirable to find the general continuous solution of equation (132) without assuming a monotonicity of $g(x)$. In the complex domain equation (132) was treated by P. J. Myrberg [2] in the case where $g(x)$ is a rational function.

The equation

$$\varphi^n(x)=g[\varphi^m(x)], \qquad m<n, \tag{133}$$

may be reduced to (132) (Kuczma [14]). In the case where $g(x)=x$ the general continuous solution of equation (133) has been given by G. M. Ewing-W. R. Utz [1].

The special case of (132)

$$\varphi^2(x)=g(x) \tag{134}$$

is of a particular interest. It is an open question under what conditions on $g(x)$ equation (134) (or, more generally, equation (132)) has a convex solution and whether such a solution is unique. This problem may be put into a more general form: under what conditions on $g(x)$ there exists an iteration group $g^u(x)$ such that $g^u(x)$ is convex for $u>0$ and concave for $u<0$? Is such a group unique?

In the complex domain equation (134) with an entire function $g(x)$ was treated by W. J. Thron [1] and I. N. Baker [1], [2]. Especially the case $g(x)=e^{x-1}$ has attracted attention of several authors (e.g. Thron [1], Osserman [1], Baker [2]). I. N. Baker [2] showed that there can exist no solution of the equation $\varphi^2(x)=e^{x-1}$ analytic at $x=0$.

Similarly the equation

$$\varphi^2(x)=e^x \tag{135}$$

is of interest. A treatment of this equation in the real case is especially difficult, since e^x has no real fixed-points (i.e. equation $e^x=x$ has no real solutions). H. Kneser [1] succeeded in obtaining a solution of equation (135) which is real and analytic on the real axis. This solution is not single-valued (Baker [1]) and, as pointed out by G. Szekeres [3], there is no uniqueness attached to the solution. G. Szekeres [3] proposes another approach to this problem; he obtains a solution of (135) as the member $g^{\frac{1}{2}}(x)$ of the regular iteration group $g^u(x)$ of $g(x)=e^x$ (cf. § 17).

Some other equations with implication index equal n were considered by J. Heinhold [1] and M. Bajraktarević [9].

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