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On monotonic solutions of a functional equation (I)

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The object of the present paper is the functional equation

$$\varphi[\varphi(x)] = g(x, \varphi(x)), \tag{1}$$

where $\varphi(x)$ is the unknown function and $g(x, y)$ is given. In the case where the function $g(x, y)$ does not depend on x equation (1) has been solved in my paper [1]. The purpose of the present note is to prove that under suitable conditions equation (1) possesses infinitely many solutions that are continuous and strictly increasing in a certain interval.

We shall assume the following hypotheses regarding the function $g(x, y)$:

(i) $g(x, x) > x$ in an interval $\langle x_0, b)$, $g(b, b) = b$.

(ii) $g(x, y)$ is continuous and strictly increasing with respect to each variable in the closure $\overline{T}$ of the set

$$T: \quad x_0 \leqslant x < b, \; x < y < \beta(x),$$

where the function $\beta(x)$ fulfils the conditions

$$g(x, y) > y \quad \text{for} \quad y \in \langle x, \beta(x)), \qquad g(x, \beta(x)) = \beta(x), \qquad x \in \langle x_0, b) \tag{2}$$

[1].

From the monotonity of the function $g(x, y)$ it follows that the function $\beta(x)$ is increasing and

$$\beta(x) \leqslant b \quad \text{for} \quad x \in \langle x_0, b\rangle. \tag{3}$$

The result of this paper is given by the following

THEOREM. *Under hypotheses* (i), (ii) *equation* (1) *possesses infinitely many solutions that are continuous and strictly increasing in the interval* $\langle x_0, b\rangle$.

Proof. Let x_1 be an arbitrary point from the interval $(x_0, \beta(x_0))$

[1] We admit also the case $b = \infty$ as well as $\beta(x) \equiv \infty$. Continuity is then understood as the existence of a suitable (infinite) limit.

and let the sequence x_n be defined for $n > 1$ by the formula

(4) $$x_{n+2} = g(x_n, x_{n+1}).$$

We shall show that

(5) $$(x_n, x_{n+1}) \in T \quad \text{for} \quad n = 0, 1, 2, \ldots$$

For $n = 0$ it is evident. Let us suppose that (5) holds true for a certain $n \geqslant 0$. On account of (2) and of the monotonity of the function $g(x, y)$ we have

(6) $$y < g(x, y) < \beta(x) \quad \text{for} \quad y \in \langle x, \beta(x)\big).$$

Consequently, by (3) and (5)

$$x_0 < x_{n+1} < x_{n+2} < \beta(x_n) \leqslant \beta(x_{n+2}) \leqslant b,$$

which proves that $(x_{n+1}, x_{n+2}) \in T$.

Thus the sequence x_n is strictly increasing and then convergent:

$$x_n \to \bar{x}.$$

Passing to a limit in relation (4) we obtain $\bar{x} = g(\bar{x}, \bar{x})$, whence it follows that $\bar{x} = b$.

Now let $\varphi_0(x)$ be an arbitrary function continuous and strictly increasing in the interval $\langle x_0, x_1\rangle$ and fulfilling the conditions

$$\varphi_0(x_0) = x_1, \quad \varphi_0(x_1) = x_2,$$
$$\big(x, \varphi_0(x)\big) \in T \quad \text{for} \quad x \in (x_0, x_1).$$

We put

(7) $$\varphi_{n+1}(x) \overset{\mathrm{df}}{=} g\big(\varphi_n^{-1}(x), x\big) \quad \text{for} \quad x \in \langle x_{n+1}, x_{n+2}\rangle.$$

We shall prove that for every n the function $\varphi_n(x)$ is continuous and strictly increasing in the interval $\langle x_n, x_{n+1}\rangle$ and

(8) $$\varphi_n(x_n) = x_{n+1}, \quad \varphi_n(x_{n+1}) = x_{n+2},$$

(9) $$\big(x, \varphi_n(x)\big) \in T \quad \text{for} \quad x \in (x_n, x_{n+1}).$$

For $n = 0$ it is so by the hypothesis. Let us suppose it true for a certain $n \geqslant 0$. Then the function $\varphi_n(x)$ is invertible in the interval $\langle x_{n+1}, x_{n+2}\rangle$ and

$$\varphi_n^{-1}(x) \in \langle x_n, x_{n+1}\rangle \quad \text{for} \quad x \in \langle x_{n+1}, x_{n+2}\rangle.$$

From (8) and (9) it follows that

$$\big(\varphi_n^{-1}(x), x\big) \in T \quad \text{for} \quad x \in \langle x_{n+1}, x_{n+2}\rangle.$$

Thus the function $\varphi_{n+1}(x)$ is, by formula (7), defined for $x \in \langle x_{n+1}, x_{n+2}\rangle$. Its continuity and monotonity follow from the continuity and monotonity of the functions $g(x, y)$ and $\varphi_n(x)$. Further,

(10) $$x_0 < x_{n+1} \leqslant x \leqslant x_{n+2} < b.$$

Moreover, according to (9),

$$x = \varphi_n[\varphi_n^{-1}(x)] < \beta[\varphi_n^{-1}(x)],$$

whence by (6) and (7)

(11) $$x < \varphi_{n+1}(x) < \beta[\varphi_n^{-1}(x)] \leqslant \beta(x).$$

From relations (10) and (11) it follows that

$$\big(x, \varphi_{n+1}(x)\big) \in T \quad \text{for} \quad x \in \langle x_{n+1}, x_{n+2}\rangle.$$

Furthermore, by (8)

$$\varphi_{n+1}(x_{n+1}) = g[\varphi_n^{-1}(x_{n+1}), x_{n+1}] = g(x_n, x_{n+1}) = x_{n+2},$$
$$\varphi_{n+1}(x_{n+2}) = g[\varphi_n^{-1}(x_{n+2}), x_{n+2}] = g(x_{n+1}, x_{n+2}) = x_{n+3}.$$

Now if we put

(12) $$\varphi(x) = \begin{cases} \varphi_n(x) & \text{for} \quad x \in \langle x_n, x_{n+1}), \quad n = 0, 1, 2, \ldots, \\ b & \text{for} \quad x = b, \end{cases}$$

then, as it can easily be verified, the function $\varphi(x)$ is defined, continuous and strictly increasing in the interval $\langle x_0, b\rangle$. We shall show that it satisfies equation (1).

Let us take an arbitrary $x \in \langle x_0, b)$. There exists an n such that $x \in \langle x_n, x_{n+1})$. Thus $\varphi(x) \in \langle x_{n+1}, x_{n+2})$. We have by (12) and (7)

$$\varphi(x) = \varphi_n(x),$$

$$\varphi[\varphi(x)] = \varphi_{n+1}[\varphi_n(x)] = g\big(\varphi_n^{-1}[\varphi_n(x)], \varphi_n(x)\big) = g\big(x, \varphi_n(x)\big) = g\big(x, \varphi(x)\big).$$

From the relation $g(b, b) = b$ it follows immediately that also for $x = b$ the function $\varphi(x)$ satisfies equation (1).

Since the value x_1 and the function $\varphi_0(x)$ can be chosen in infinitely many ways we obtain thus infinitely many solutions.

Remark. Equation (1) is a particular case of the equation

(13) $$\varphi\big[f\big(x, \varphi(x)\big)\big] = g\big(x, \varphi(x)\big).$$

However, if the function $f(x, y)$ is invertible with respect to y, equation (13) can be reduced to equation (1) by the substitution

$$\psi(x) = f\big(x, \varphi(x)\big).$$

Reference

[1] M. Kuczma, *On some functional equations containing iterations of the unknown function*, Ann. Polon. Math. (in press).

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