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Iterative Functional Equations

MAREK KUCZMA
Institute of Mathematics, Silesian University, Katowice, Poland

BOGDAN CHOCZEWSKI
Institute of Mathematics, Academy of Mining and Metallurgy, Kraków, Poland

ROMAN GER
Institute of Mathematics, Silesian University, Katowice, Poland

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all r_0. On the other hand, it is not generally true that if, for every $r < \infty$, the C^r solution of equation (3.4.1) depends on an arbitrary function, then so does the C^∞ solution (see Note 3.8.10).

Comments. Theorem 3.4.2 is the first one in this theory on differentiable solutions. It has been proved with the aid of Banach's Theorem (Kuczma [7]). (For a use of topological methods in the theory of iterative functional equations see Choczewski [7].) Fixed-point theorems, in general, cannot be applied in the indeterminate case ($|g(0)|(f'(0))^r = 1$) for C^r solutions. The results concerning this case are very scarce (see Choczewski [4], [6], Czerwik [5], [12]).

Linear equations have also been studied in the class of functions whose rth derivative is either Lipschitzian (Jelonek [1]) or absolutely continuous (Sieczko [1]).

3.5 Special equations

In this section we are going to apply some results from the preceding sections to the problem of uniqueness of solutions to the equations of Schröder, Abel and Julia. The first two of them have already been studied in Chapter 2, but in other function classes, and they are more thoroughly treated in Chapters 8 and 9. The reader is referred to the latter chapter for notes and further applications of the Schröder and Abel equations. The Julia equation

$$\lambda(f(x)) = f'(x)\lambda(x)$$

(Julia [1]) plays an important role in the theory of continuous iteration (see e.g. Ecalle [1], Dubuc [1]) but this aspect exceeds the scope of our book. In Chapter 8 Julia's equation is used to determine conjugate and permutable power series.

3.5A. Schröder's equation

We aim at proving a uniqueness theorem for differentiable solutions $\sigma: X \to \mathbb{R}$ of the equation

$$\sigma(f(x)) = s\sigma(x) \tag{3.5.1}$$

(Crum [1], Szekeres [1]). We assume that

(i) $X = [0, a|, 0 < a \leqslant \infty$,
(ii) $f: X \to X$ is of class C^1 in X, $0 < f(x) < x$, $f'(x) \neq 0$ in $X \setminus \{0\}$ and

$$f'(x) = s + O(x^\delta), \qquad x \to 0,\ \delta > 0,\ 0 < s < 1.$$

Remark 3.5.1. If $f \in C^2(X)$ and $f'(0) = s$, then this asymptotic relation is certainly fulfilled.

Theorem 3.5.1. *Let hypotheses* (i), (ii) *be fulfilled. Then equation* (3.5.1) *has a unique solution $\sigma: X \to \mathbb{R}$ of class C^1 on X and satisfying $\sigma'(0) = 1$. This solution is given by the formula*

$$\sigma(x) = \lim_{n \to \infty} s^{-n} f^n(x), \tag{3.5.2}$$

is strictly increasing in X, and fulfils the condition

$$\sigma'(x) = 1 + O(x^\delta), \qquad x \to 0. \tag{3.5.3}$$

Proof. First observe that $\sigma(0) = 0$ for any solution $\sigma: X \to \mathbb{R}$ of (1). Further, equation (3.5.1) has a C^1 solution σ in X such that $\sigma'(0) = 1$ if and only if the equation

$$\varphi(f(x)) = \frac{s}{f'(x)}\varphi(x) \tag{3.5.4}$$

has a continuous solution $\varphi: X \to \mathbb{R}$ such that $\varphi(0) = 1$. Clearly, $\sigma(x) = \int_0^x \varphi(t)dt$.

Since $f'(x) = s + O(x^\delta)$, we have $f(x) = sx + O(x^{1+\delta})$ and $s/f'(x) = 1 + O(x^\delta)$ as $x \to 0$. We see that Theorem 3.1.13 applies to equation (3.5.4). Consequently, a continuous solution $\varphi: X \to \mathbb{R}$ of (3.5.4) such that $\varphi(0) = 1$ does exist and is unique. Thus the same is true for the C^1 solution σ of (3.5.1) in X such that $\sigma'(0) = 1$.

To prove (3.5.3) for this σ we proceed as follows. The formula

$$\hat{\sigma}'(x) = 1 + x^\delta \hat{\phi}(x), \qquad x \in X \setminus \{0\},\ \hat{\phi}(0) = 0,$$

links C^1 solutions $\hat{\sigma}$ of (1) satisfying (3.5.3) with solutions $\hat{\phi}$ belonging to the class $\mathscr{B}$ (with $Y = \mathbb{R}$; see (3.1.23)) of the equation

$$\hat{\phi}(f(x)) = \hat{g}(x)\hat{\phi}(x) + \hat{h}(x), \tag{3.5.3}$$

where

$$\hat{g}(x) := \begin{cases} (x/f(x))^\delta s/f'(x), & x \in X \setminus \{0\}, \\ s^{-\delta}, & x = 0, \end{cases}$$

$$\hat{h}(x) := \begin{cases} (s/f'(x) - 1)^\delta (f(x))^{-\delta}, & x \in X \setminus \{0\}, \\ 0, & x = 0. \end{cases}$$

Since $|\hat{g}(0)| > 1$, from Theorem 3.1.11 we infer that equation (3.5.5) has a unique solution in the class $\mathscr{B}$. The corresponding solution $\hat{\sigma}$ of (3.5.1) is also unique. But $\hat{\sigma}$ is of class C^1 in X and $\hat{\sigma}'(0) = 1$ so that $\hat{\sigma} = \sigma$. Thus the already determined σ has property (3.5.3).

Since $\sigma'(0) = 1$, σ is strictly increasing in a neighbourhood of the origin, and by (3.5.1) it is so in X.

The last thing to be proved is formula (3.5.2). Iterating (3.5.1) we obtain by induction

$$\sigma(x) = s^{-n}\sigma(f^n(x)) = (s^{-n}f^n(x))(\sigma(f^n(x))/f^n(x))$$

for $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$. Hence (3.5.2) follows as $\lim_{n \to \infty} f^n(x) = 0$ and $\sigma'(0) = 1$.For $x = 0$ (3.5.2) is trivial. ■

3.5B. Julia's equation

We start by presenting some results by M. C. Zdun [3] on solutions of the Julia equation

$$\lambda(f(x)) = f'(x)\lambda(x) \tag{3.5.6}$$

belonging to the following function class:

$$\mathscr{D} := \{\varphi\colon X \to \mathbb{R},\ \varphi \text{ is continuous on } X \text{ and differentiable at } x = 0\}.$$

To this end assume that

(iii) $f\colon X \to X$ is convex or concave and of class C^1 on X, $0 < f(x) < x$ and $f'(x) \neq 0$ in $X\setminus\{0\}$.

In the sequel we write

$$s := f'(0).$$

By (iii) we have $0 \leqslant s \leqslant 1$.

Theorem 3.5.2. *Let hypotheses* (i), (iii) *be fulfilled. Then the solutions* $\lambda \in \mathscr{D}$ *of equation* (3.5.6) *are the following.*

(1) $0 < s < 1$. *All the solutions are given by*

$$\lambda(x) = c \lim_{n\to\infty} \frac{f^n(x)}{(f^n)'(x)}, \tag{3.5.7}$$

where $c \in \mathbb{R}$ *is an arbitrary constant (a parameter).*

(2) $s = 1$. *The solution depends on an arbitrary function and* $\lambda'(0) = 0$ *for every solution* λ.

(3) $s = 0$. *The only solution is* $\lambda = 0$.

Proof. If $s < 1$, then $\lambda(0) = 0$ for every solution λ of (3.5.6). If $s = 1$, then f must be concave, and so are all f^n, $n \in \mathbb{N}$. Hence $(f^n)'(x) \leqq f^n(x)/x$ and $\lim_{n\to\infty}(f^n)'(x) = 0$ in $X\setminus\{0\}$. From (3.5.6) we get

$$\lambda(f^n(x)) = \prod_{i=0}^{n-1} f'(f^i(x))\lambda(x) = (f^n)'(x)\lambda(x), \qquad n \in \mathbb{N},$$

whence $\lambda(0) = 0$ for every continuous solution λ of (3.5.6). Thus $\lambda \in \mathscr{D}$ is a solution of (3.5.6) if and only if the function $\varphi(x) = \lambda(x)/x$ ($\varphi(0) = \lambda'(0)$) is a continuous solution of the equation

$$\varphi(f(x)) = g(x)\varphi(x), \tag{3.5.8}$$

where $g(x) := xf'(x)/f(x)$ for $x \in X\setminus\{0\}$ and $g(0) := 1$. Then the function $g\colon X \to \mathbb{R}$ is continuous in X and $g \geqq 1$ or $g \leqq 1$ according as f is convex or concave. Therefore the sequence (G_n) defined by (3.1.4) is increasing or decreasing according as f is convex or concave.

If f is convex, then from the relation

$$\int_{f(x)}^{x} f'(t)\mathrm{d}t = f(x) - f^2(x)$$

we get the estimation

$$(x - f(x))f'(f(x)) \leqq f(x) - f^2(x) \leqq (x - f(x))f'(x),$$

whence by induction

$$\frac{x - f(x)}{f'(x)} \prod_{i=0}^{n} f'(f^i(x)) \leqq f^n(x) - f^{n+1}(x) \leqq (x - f(x)) \prod_{i=0}^{n-1} f'(f^i(x))$$

for $x \in X\setminus\{0\}$, $n \in \mathbb{N}$. By the equality

$$G_n(x) = \frac{x}{f^n(x)} \prod_{i=0}^{n-1} f'(f^i(x))$$

this may be rewritten as

$$G_{n+1}(x)\frac{f^{n+1}(x)}{f^n(x)f'(x)} \leqq \frac{x(1 - f^{n+1}(x)/f^n(x))}{x - f(x)} \leqq G_n(x). \tag{3.5.9}$$

If f is concave, then the inequalities in (3.5.9) are reversed.

Assume that $0 < s < 1$. Since the function $x \mapsto f(x)/x$, $x \in X\setminus\{0\}$ is monotonic and approaches s as $x \to 0$, we get from (3.5.9)

$$1 \leqq G_{n+1}(x) \leqq h(x) \tag{3.5.10}$$

for the case where f is convex; and (3.5.10) with the inequalities reversed if f is concave. Here we have put

$$h(x) = \frac{(1 - s)xf'(x)}{s(x - f(x))}$$

so that $\lim_{x\to 0} h(x) = 1$. Thus we have by (3.5.10), in view of the monotonicity of the sequence (G_n),

$$|G_{n+p}(x) - G_n(x)| = G_n(x)|G_p(f^n(x)) - 1| \leqq \max(h(x), 1)|h(f^n(x)) - 1|$$

for $n, p \in \mathbb{N}$, $x \in X$. This implies (Theorem 1.2.4) that the sequence (G_n) converges a.u. in X to a continuous function vanishing nowhere on X. By Theorem 3.1.2 equation (3.5.8) has in X a unique one-parameter family of continuous solutions:

$$\varphi(x) = c\Big/\lim_{n\to\infty} G_n(x) = \frac{c}{x}\lim_{n\to\infty}(f^n(x)/(f^n)'(x)), \quad x \neq 0; \qquad \varphi(0) = c. \tag{3.5.11}$$

This proves part (1) of the theorem.

If $s = 1$, then f must be concave and it follows from (3.5.9) (inequalities reversed!) that $\lim_{n\to\infty} G_n(x) = 0$ a.u. in $X\setminus\{0\}$. By Theorem 3.1.3 equation (3.5.8) has in X a continuous solution depending on an arbitrary function and every such solution vanishes at the origin (Corollary 3.1.1). This gives the conclusion of part (2).

Finally, let $s=0$ and suppose equation (3.5.1) has a nontrivial solution $\lambda \in \mathcal{D}$. The function f must be convex and so it follows by (3.5.9) that the sequence (G_n) is bounded away from zero in X. Thus necessarily case (A) occurs for equation (3.5.8). Through (3.5.11) we come to formula (3.5.7) for our λ, with a $c \neq 0$. Moreover, $\lambda(x) \neq 0$ in $X \setminus \{0\}$, by the argument we have used in the proof of Theorem 3.1.4. Since $f' \circ f^n \geqslant f^{n+1}/f^n$, the sequence $((f^n)'/f^n)$ is increasing and we get from (3.5.7) the inequality $0 < (f^n)'(x)/f^n(x) < c/\lambda(x)$ for $x \in X \setminus \{0\}$, whence

$$\int_{f(x)}^{x} \frac{c\,dt}{\lambda(t)} \geqslant \int_{f(x)}^{x} \frac{(f^n)'(t)}{f^n(t)}\,dt = \log \frac{f^n(x)}{f^{n+1}(x)} \to \infty$$

as $n \to \infty$, which is impossible. Thus $\lambda = 0$. ■

The following theorem shows a connection between the equations of Julia and Schröder.

Theorem 3.5.3. *Let hypotheses* (i) *and* (iii) *with* $0<s<1$ *be fulfilled. Let* $\sigma: X \to \mathbb{R}$ *be a nontrivial convex or concave solution of equation* (3.5.1) *and let* $\lambda \in \mathcal{D}$ *be a solution of equation* (3.5.6). *Then* σ *is of class* C^1 *in* $X \setminus \{0\}$ *(and even in* X *if* $\lim_{x \to 0} \sigma'(x) < \infty$), *and the functions* λ *and* σ *are related by the formula*

$$\lambda(x) = \lambda'(0)\sigma(x)/\sigma'(x) \qquad \textit{in } X \setminus \{0\}. \tag{3.5.12}$$

Proof. The solutions λ and σ exist in virtue of Theorems 3.5.2 and 2.4.4. The case of $\lambda = 0$ is obvious, so assume $\lambda'(0) \neq 0$. Then $\lambda(x) \neq 0$ in $X \setminus \{0\}$ (see the proof of part (1) of Theorem 3.5.2) and $\lambda'(0)/\lambda(x) = \lim_{n\to\infty}((f^n)'(x)/f^n(x))$ a.u. in $X \setminus \{0\}$. By integrating this relation we have for arbitrary x, x_0 from $X \setminus \{0\}$

$$\lambda'(0)\int_{x_0}^{x} \frac{dt}{\lambda(t)} = \lim_{n\to\infty} \log \frac{f^n(x)}{f^n(x_0)} = \log\left(\frac{1}{c}\sigma(x)\right)$$

(see formula (2.4.18)), whence

$$\sigma(x) = c \exp\left[\lambda'(0)\int_{x_0}^{x} (\lambda(t))^{-1}dt\right]. \tag{3.5.13}$$

It follows from (3.5.13) that σ is of class C^1 in $X \setminus \{0\}$ and that (3.5.12) holds. Since σ is convex or concave, there exists $\sigma'(0) = \lim_{x\to 0}\sigma'(x)$. If this limit is finite, then σ is of class C^1 in X. ■

Finally, with the aid of Theorem 3.3.4 we shall prove the following theorem which is implicitly contained in a result by G. Szekeres [1].

Theorem 3.5.4. *Make the assumptions* (i) *and* (ii), *but with the asymptotic relation replaced by*

$$f'(x) = 1 - b(m+1)x^m + O(x^{m+\delta}), \qquad x \to 0, \tag{3.5.14}$$

where b, m, δ *are some positive constants. Then equation* (3.5.6) *has a uniq one-parameter family of continuous solutions* $\lambda: X \to \mathbb{R}$ *such that*

$$\lambda(x) = x^{m+1}(c + O(x^\tau)), \qquad x \to 0,\ c \in \mathbb{R}, \qquad (3.5.1$$

where $\tau = \min(m, \delta)$. *They are given by the formula*

$$\lambda(x) = c \lim_{n\to\infty} [(f^n(x))^{m+1}/(f^n)'(x)]. \qquad (3.5.1$$

Proof. Let us put

$$\lambda(x) = x^{m+1}\varphi(x). \qquad (3.5.1$$

To each solution $\lambda: X \to \mathbb{R}$ of equation (3.5.6) having the properties stat in the theorem there corresponds a continuous solution $\varphi: X \to \mathbb{R}$ such th $\varphi(x) = c + \sigma(x^\tau)$, $x \to 0$, of equation (3.5.8) where

$$g(x) := x^{m+1}f'(x)(f(x))^{-m-1}, \qquad x \in X \setminus \{0\}, \qquad g(0) := 1,$$

and conversely. The two solutions are linked by (3.5.17).

To solve equation (3.5.8) with our g, note that relation (3.5.14) impli

$$f(x) = x - bx^{m+1} + O(x^{m+1+\delta}). \qquad (3.5.1$$

Consequently, by the definition of g,

$$g(x) = 1 + O(x^{m+\tau}), \qquad x \to 0.$$

By Theorem 3.3.4 (see also Theorem 3.1.2) equation (3.5.8) has a uniq one-parameter family of continuous solutions $\varphi: X \to \mathbb{R}$. They are given by

$$\varphi(x) = c \lim_{n\to\infty} \prod_{i=0}^{n-1} (g(f^i(x)))^{-1} = c \lim_{n\to\infty} [x^{-m-1}(f^n(x))^{m+1}/(f^n)'(x)]$$

and they have the property $\varphi(x) = c + O(x^\tau)$, $x \to 0$. The functions λ defin by (3.5.17) fulfil the conditions of the theorem. ■

3.5C. Abel's equation

The equation is of the form

$$\alpha(f(x)) = \alpha(x) + 1. \qquad (3.5.1$$

Clearly, equation (3.5.19) cannot have a solution defined at the fixed poi of f. Therefore now $0 \notin X$. We then replace hypothesis (i) by

(i′) $X = (0, a|$, $0 < a \leqq \infty$.

We aim at showing that if $0 < s < 1$ in (ii) then some solutions of equatio (3.5.19) can be obtained via Theorem 3.5.1 with the aid of differentiab solutions of a Schröder equation.

Theorem 3.5.5. *Assume* (i′) *and* (ii). *Then equation* (3.5.19) *has a uniq one-parameter family (with an additive parameter) of solutions* $\alpha: X \to \mathbb{R}$ *whic* *fulfil the condition*

$$\alpha(x) = \log x/\log s + \varphi(x)$$

where $\lim_{x\to 0} \varphi(x)$ *exists and is finite. These solutions are given by the formula*

$$\alpha(x) = \log \sigma(x)/\log s + c, \qquad c \in \mathbb{R}, \tag{3.5.20}$$

where $\sigma: X \cup \{0\} \to \mathbb{R}$ *is a* C^1 *solution of the Schröder equation* (3.5.1) *in* $X \cup \{0\}$ *such that* $\sigma'(0) = 1$.

Proof. We can extend f onto $X \cup \{0\}$ by putting $f(0) = 0$. By Theorem 3.5.1 equation (3.5.1) has a unique C^1 solution $\sigma: X \cup \{0\} \to \mathbb{R}$ fulfilling the condition $\sigma'(0) = 1$. It is easily seen that α given by (3.5.20) with $c = 0$ has all the required properties.

Now let $\tilde{\alpha}: X \to \mathbb{R}$ be another solution of equation (3.5.19) satisfying $\tilde{\alpha}(x) = \log x/\log s + \tilde{\varphi}(x)$, where $\tilde{\varphi}$ approaches a finite limit as $x \to 0$. Then the function $\omega(x) := \tilde{\alpha}(x) - \alpha(x) = \tilde{\varphi}(x) - \varphi(x)$ satisfies the equation $\omega(f(x)) = \omega(x)$ and has a finite limit at zero, whence $\omega = \text{const}$. Thus the solution α is unique up to an additive constant. ■

Differentiating both sides of (3.5.19) we get

$$\alpha'(f(x))f'(x) = \alpha'(x).$$

Thus the Julia equation is related also to that of Abel. In the case where $s = 1$ we have the following theorem (Szekeres [1]).

Theorem 3.5.6. *Assume* (i′) *and* (ii) *but with the asymptotic relation replaced by* (3.5.14). *Then equation* (3.5.19) *has a unique one-parameter family of solutions* $\alpha: X \to \mathbb{R}$ *which are of class* C^1 *in* X *and fulfil the condition*

$$\alpha'(x) = x^{-m-1}\varphi(x), \tag{3.5.21}$$

where $\lim_{x\to 0} \varphi(x)$ *exists and is finite. These solutions are strictly decreasing in* X, *and are given by the formula (of Lévy; see Remark 9.1.2)*

$$\alpha(x) = c + \lim_{n\to\infty} \frac{f^n(x) - f^n(x_0)}{f^{n+1}(x_0) - f^n(x_0)}, \tag{3.5.22}$$

where $x_0 \in X$ *is arbitrarily fixed and* c *is an arbitrary constant (a parameter). Moreover*

$$\alpha'(x) = -b^{-1}x^{-m-1} + O(x^{-m-1+\tau}), \qquad x \to 0, \tag{3.5.23}$$

where $\tau = \min(m, \delta)$.

Proof. This will be sketched only. Fix an $x_0 \in X$ and put

$$\alpha(x) = r\int_{x_0}^{x} \frac{dt}{\lambda(t)} \tag{3.5.24}$$

where $\lambda: X \to \mathbb{R}$ is given by (3.5.16) with $c = 1$ and has the properties stated in Theorem 3.5.4. The constant r is to be determined.

The following facts can be verified:

- the convergence in (3.5.16) is almost uniform in X,
- by Theorem 1.3.6, (3.5.14) and the monotonicity of f^n

$$\lim_{n\to\infty} (nbm)^{1/m} f^n(x) = 1 \qquad (3.5.2$$

- by (3.5.25), formula (3.5.16) goes over into

$$\lambda(x) = \lim_{n\to\infty} (nbm)^{-1-1/m}/(f^n)'(x) \qquad \text{a.u. in } X,$$

  so that interchanging 'lim' with '∫' signs in (3.5.24) yields

$$\alpha(x) = r \lim_{n\to\infty} (nbm)^{1+1/m}(f^n(x) - f^n(x_0)), \qquad (3.5.2$$

- since λ satisfies (3.5.6), we have $\alpha(f(x)) - \alpha(x) = \alpha(f(x_0))$,
- by (3.5.18) (see (3.5.14)), (3.5.26) and (3.5.25) yield $\alpha(f(x_0)) = -r$
- taking in (3.5.26) $r = -1/b$ we get (3.5.19),
- formula (3.5.22) with $c = 0$ is obtained by using (3.5.26) ($r = -1/b$ for both $\alpha(x)$ and $\alpha(f(x_0)) = 1$, and then forming their quotien
- the regularity and monotonicity of α follow from (3.5.24) ($r = -1/b$
- relation (3.5.23) is a consequence of (3.5.21) that results fro (3.5.24) ($r = -1/b$) and (3.5.15) ($c = 1$).

To prove the uniqueness, observe that if $\hat{\alpha}$ is a C^1 solution of (3.5.19) i X satisfying (3.5.21), then $\hat{\lambda} = 1/\hat{\alpha}'$ (defined as 0 at the origin) must be continuous solution of equation (3.5.6) in $X \cup \{0\}$, fulfilling conditio (3.5.15). By Theorem 3.5.4, $\lambda = q\hat{\lambda}$. Thus $\hat{\alpha}(x) = (q/r)(c + \alpha(x))$ and $q = r$ as satisfies (3.5.19). ■

3.5D. A characterization of the cross ratio

Let us put

$$D := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_i \neq x_j \text{ for } i \neq j\}.$$

The cross ratio $s: D \to \mathbb{R}$ of four points of the projective line is given by th formula

$$s(x_1, x_2, x_3, x_4) = (x_1 - x_3)(x_2 - x_4)/(x_2 - x_3)(x_1 - x_4).$$

S. Gołąb [2] considered the following system of functional equations:

$$S(x_3, x_4, x_1, x_2) = S(x_1, x_2, x_3, x_4), \qquad (3.5.2$$

$$S(x_1, x_3, x_2, x_4) + S(x_1, x_2, x_3, x_4) = 1, \qquad (3.5.2$$

$$S(x_1, x_2, x_3, x_4) \cdot S(x_1, x_2, x_4, x_5) = S(x_1, x_2, x_3, x_5). \qquad (3.5.2$$

Of course, $S = s$ satisfies (3.5.27)–(3.5.29), but the general solution of th system, without any regularity assumptions, is given by (see Gołąb [2])

$$S(x_1, x_2, x_3, x_4) = s(\gamma(x_1), \gamma(x_2), \gamma(x_3), \gamma(x_4)), \qquad (3.5.3$$

where $\gamma: \mathbb{R} \to \mathbb{R}$ is an arbitrary injection on $\mathbb{R}$.

To get a characterization of the cross ratio use can be made of the conditio

$S = s$ imposed on a one-parameter family of harmonic points of the projective line, i.e.,

$$S\left(x, 1, \frac{2x}{x+1}, 0\right) = -1, \qquad x \in \mathbb{R}' := \mathbb{R} \setminus \{-1, 0, 1\}. \tag{3.5.31}$$

Condition (3.5.31) leads to a functional equation for γ which in turn can be transformed to the Schröder equation

$$\sigma\left(\frac{x}{x+2}\right) = \tfrac{1}{2}\sigma(x) \qquad x \in \mathbb{R}'' := \mathbb{R} \setminus \{-2, -1, 0\}. \tag{3.5.32}$$

If $\sigma: \mathbb{R} \to \mathbb{R}$ is an injection satisfying (3.5.32), then the function S given by (3.5.30) with γ defined by

$$\gamma(x) := b + (\sigma(x+1) - \sigma(1))^{-1}, \qquad x \neq 0, \ \gamma(0) := b \tag{3.5.33}$$

(b may be arbitrary) has property (3.5.31). The converse is also true.

The injections $\sigma: \mathbb{R} \to \mathbb{R}$ given by

$$\sigma(x) = c\frac{x}{x+1}, \quad x \neq -1, \ \sigma(-1) = c, \ c \in \mathbb{R}, \tag{3.5.34}$$

satisfy equation (3.5.32). Using them in (3.5.33) we get homographies, too, and $S = s$ with such γs. Since the function $f: X \to X$, $X := [0, a)$, $a > 0$, $f(x) = x/(x+2)$ satisfies the assumption of Theorem 3.5.1 (in particular, $f'(x) = \frac{1}{2} + O(x)$, $x \to 0$), formula (3.5.34) yields all solutions $\sigma \in C^1(X)$ of (3.5.32). Thus we have the following (Choczewski [13]).

Theorem 3.5.7. *Let $S: D \to \mathbb{R}$ be a function given by* (3.5.30) *with a $\gamma: \mathbb{R} \to \mathbb{R}$ injective and of class C^1 in a neighbourhood of $x = 1$. If S satisfies* (3.5.31), *then $S = s$. In other words: equations* (3.5.27), (3.5.28), (3.5.29) *and* (3.5.31) *then characterize the cross ratio s.*

3.6 Solutions of bounded variation

The theory of linear equations in this function class is also rather widely developed. An extensive study of solutions of bounded variation has been carried out by M. C. Zdun [4], [5], [7], [8], [15] (see also Matkowski–Zdun [1], Lasota [1]). Here we present a few of Zdun's results.

3.6A. Preliminaries

We denote by $\operatorname{Var} \varphi | X$ the *variation* of the function $\varphi: X \to \mathbb{R}$ on the interval X. If X is not compact, then $\operatorname{Var} \varphi | X$ is meant as the supremum of $\operatorname{Var} \varphi | I$ taken over all compact subintervals I of X.

We shall look for solutions in the class of functions

$$\mathrm{BV}X := \{\varphi: X \to \mathbb{R}, \ \operatorname{Var} \varphi | X < \infty\}$$

that are of *bounded variation* on X. For the homogeneous equation

$$\varphi(f(x)) = g(x)\varphi(x) \tag{3.6.}$$

we make the following hypotheses.

(i) $X = (0, a|$, $0 < a \leqslant \infty$.
(ii) $f: X \to \mathbb{R}$ is continuous and strictly increasing, $0 < f(x) < x$ in
(iii) $g \in \mathrm{BV}X$ and $m := \inf_X g > 0$.

As usual, the sequence $(G_n)_{n \in \mathbb{N}_0}$ is defined by (3.1.4). Moreover, througho this section we use the following notations ($x_0 \in X$, $i \in \mathbb{N}_0$):

$$X_i = [f^{i+1}(x_0), f^i(x_0)];$$
$$a_i = \sup_{X_i} g = \sup_{X_0} g \circ f^i;$$
$$b_i = \sup_{X_0} G_i;$$
$$v_i = \operatorname{Var} g | X_i = \operatorname{Var} g \circ f^i | X_0;$$
$$A_i = \prod_{j=0}^{i-1} a_j.$$

The lemma that follows shows the role of the sequence (G_n).

Lemma 3.6.1. *Under hypotheses* (i)–(iii) *we have*

$$0 < L^{-1}G_n(x_1) \leqq A_n \leqq LG_n(x_2) \qquad \text{for } x_1, x_2 \in X_0, \ n \in \mathbb{N}_0, \tag{3.6.}$$

where

$$L := \exp \operatorname{Var} \log g | X \tag{3.6.}$$

does not depend on x_0; and

$$\operatorname{Var} G_n | X_0 \leqq KG_n(x_0), \qquad n \in \mathbb{N}_0, \tag{3.6.}$$

where K is a positive constant independent of x_0.

Proof. We have for $x \in X_0$, $n \in \mathbb{N}_0$,

$$\left|\log \frac{A_n}{G_n(x)}\right| \leqq \sum_{i=0}^{n-1} |\log a_i - \log g(f^i(x))| \leqq \operatorname{Var} \log g | (0, x_0],$$

whence, according to (3.6.3), $L^{-1} \leqq A_n / G_n(x) \leqq L$ and (3.6.2) follows.

Further, $G_{n+1}(x) = g(f^n(x))G_n(x)$, by (3.1.4). Therefore

$$\operatorname{Var} G_{n+1} | X_0 \leqq a_n \operatorname{Var} G_n | X_0 + v_n b_n, \qquad n \in \mathbb{N}_0,$$

whence, first by induction and next by the inequality $A_{i+1} \geqq b_{i+1}$,

$$\operatorname{Var} G_n | X_0 \leqq A_n \sum_{i=0}^{n-1} \frac{v_i b_i}{A_{i+1}} \leqq A_n \sum_{i=1}^{n-1} \frac{b_i}{b_{i+1}} v_i, \qquad n \in \mathbb{N}.$$

Take in the inequalities of (3.6.2) first $x_1 = x$, $x_2 = x_0$ and then $x_1 = x_0$, $x_2 = x$ (where $x \in X_0$) to get the inequalities

$$L^{-2}G_n(x_0) \leqq G_n(x)) \leqq L^2 G_n(x_0), \qquad x \in X_0, \ n \in \mathbb{N}_0. \tag{3.6.5}$$

Theorem 4.1.2. *Make assumptions (i')–(vi'). If, moreover,*

$$|g_0(\xi)| < 1,$$

then, for $m = 0$ and for m sufficiently large, equation (4.1.3$_m$) has a unique continuous solution $\varphi_m: X \to Y$, and $\lim_{m\to\infty} \varphi_m = \varphi_0$ a.u. in X.

Proof. By (ii') and (v') there are an $m' \in \mathbb{N}$, constants $L > 0$, $\Theta \in (0, 1)$ and a compact neighbourhood C' of ξ (independent of m) such that

$$|g_m(x)| \leqq \Theta, \qquad \|h_m(x)\| \leqq L$$

for $m \geqslant m'$ and $x \in C'$. In virtue of (iv') we now choose a C such that $f_m(C) \subset C \subset C'$, $m \in \mathbb{N}_0$. The existence and uniqueness of a continuous solution $\varphi_m: X \to Y$ to equation (4.1.3$_m$) for $m = 0$ and $m \geqq m'$ result from Theorem 4.1.1 (with $X = C$) and Remark 4.1.1. By Theorem 1.6.4 we have $\lim_{m\to\infty} \varphi_m = \varphi_0$ uniformly on C.

Fix a $p \in \mathbb{N}$. It follows from (iii') that there exists an $n \in \mathbb{N}$ such that $f_0^n(\mathrm{cl}\, V_p) \subset \mathrm{int}\, C$, and hence, by (v'), $f_m^n(\mathrm{cl}\, V_p) \subset C$ for $m = 0$ and m large enough. Since the φ_m satisfy (4.1.3$_m$), we have (induction)

$$\varphi_m(x) = \varphi_m(f_m^n(x)) \prod_{i=0}^{n-1} g_m(f_m^i(x)) + \sum_{i=0}^{n-1} \left[\prod_{j=0}^{i-1} g_m(f_m^j(x))\right] h_m(f_m^i(x)). \quad (4.1.4)$$

Using formula (4.1.4) we can easily check that $\lim_{m\to\infty} \varphi_m = \varphi_0$ uniformly on cl V_p. Hence the theorem follows in view of (vi'). ■

4.2 Analytic solutions; the case $|f'(0)| < 1$

Now we consider complex-valued functions defined on subsets of $\mathbb{C}$ – the set of complex numbers. In this and the next two sections we use abbreviation 'LAS' for 'local analytic solution (of the underlying equation) in a neighbourhood of the origin'.

4.2A. Extension theorems

The question arises whether a solution analytic in a neighbourhood of the origin can be extended to a solution on a larger set with the analyticity preserved. We present two theorems to this effect which are due to J. Matkowski [2]. The equation in question is

$$p(x)\varphi(x) = g(x)\varphi(f(x)) + h(x). \quad (4.2.1)$$

Assume the following.

(i) $X \subset \mathbb{C}$ is an open set and $0 \in X$. The boundary of X contains at least two finite points.

(ii) $f: X \to X$ is an analytic function, $f(0) = 0$, $|f'(0)| < 1$, and $f(x) \neq x$ for $x \in X \setminus \{0\}$.

(iii) The functions $p: X \to \mathbb{C}$, $g: X \to \mathbb{C}$ and $h: X \to \mathbb{C}$ are analytic in X, $p(x) \neq 0$ in X.

Theorem 4.2.1. *Let hypotheses (i)–(iii) be fulfilled and let $U \subset X$ be a neighbourhood of the origin such that $f(U) \subset U$. If an analytic function $\varphi_0: U \to \mathbb{C}$ satisfies equation (4.2.1) in U, then there exists a unique solution $\varphi: X \to \mathbb{C}$ of equation (4.2.1) in X such that*

$$\varphi(x) = \varphi_0(x) \qquad \text{for } x \in U. \quad (4.2.2)$$

This function φ is analytic in X.

Proof. Take an arbitrary compact set $C_0 \subset X$ and write $C_n = f^n(C_0)$, $n \in \mathbb{N}$. By Theorem 1.2.6 we have $C_N \subset U$ for an index $N \in \mathbb{N}$. Define the functions $\varphi_i: C_{N-i} \to \mathbb{C}$ by the recurrence

$$\varphi_{i+1}(x) = (g(x)\varphi_i(f(x)) + h(x))/p(x), \qquad x \in C_{N-i-1}, \quad (4.2.3)$$

where φ_0 is the given solution of (4.2.1). By (4.2.3), φ_i is analytic in C_{N-i}, in particular so is φ_N in C_0. Denote this φ_N by $\varphi(C_0;\cdot)$. Since φ_0 satisfies equation (4.2.1) in U, the function $\varphi(C_0;\cdot)$ is independent of the choice of N.

Let K_1, K_2 be compact sets, $K_1 \subset K_2 \subset X$. We claim that

$$\varphi(K_1; x) = \varphi(K_2; x) \qquad \text{for } x \in K_1. \quad (4.2.4)$$

Indeed, if $N \in \mathbb{N}$ is such that $f^N(K_2) \subset U$, then also $f^N(K_1) \subset f^N(K_2) \subset U$. Thus $\varphi_0(K_1; x) = \varphi_0(K_2; x) = \varphi_0(x)$ for $x \in f^N(K_1)$. Making use of (4.2.3) we check that $\varphi_i(K_1; x) = \varphi_i(K_2; x)$ for $x \in f^{N-i}(K_1)$, $i = 1, \ldots, N$, and, in particular ($i = N$), $\varphi(K_1; x) = \varphi_N(K_1; x) = \varphi_N(K_2; x) = \varphi(K_2; x)$ for $x \in K_1$.

Let $(K_j)_{j\in\mathbb{N}}$ be an increasing sequence of compact sets, the union of which is X. In virtue of (4.2.4) the function $\varphi: X \to \mathbb{C}$

$$\varphi(x) = \varphi(K_j; x) \qquad \text{for } x \in K_j, j \in \mathbb{N},$$

is well defined. We shall prove that it is a solution of (4.2.1) in X. Take an $x \in X$. Since $X = \bigcup_{j=1}^{\infty} K_j$, we have $x \in K_j$ for a certain j. Further $\varphi(x) = \varphi(K_j; x) = \varphi_N(K_j; x)$ with N such that $f^N(K_j) \subset U$. Of course $f(x) \in f(K_j)$ and $f^{N-1}(f(K_j)) = f^N(K_j) \subset U$. Thus

$$\varphi(f(K_j); f(x)) = \varphi_{N-1}(f(K_j); f(x)) = \varphi_{N-1}(K_j; f(x)).$$

By (4.2.3) one gets

$$g(x)\varphi_{N-1}(K_j; f(x)) = p(x)\varphi_N(K_j; x) - h(x) = p(x)\varphi(x) - h(x). \quad (4.2.5)$$

There is an index $k \in \mathbb{N}$ such that $f(x) \in K_k$. Relation (4.2.4) yields

$$\varphi_{N-1}(K_j; f(x)) = \varphi(f(K_j); f(x)) = \varphi(f(K_j) \cup K_k; f(x)) = \varphi(K_k; f(x)) = \varphi(f(x)),$$

which together with (4.2.5) shows that (4.2.1) holds. Our claim is proved. In order to verify the relation (4.2.2) take an $x \in U$ and a compact set K such that $x \in K \subset U$. Since φ_0 satisfies (4.2.1) in U, we get by (4.2.3) $\varphi(K, x) = \varphi_0(x)$. There is a $j \in \mathbb{N}$ such that $x \in K_j$. Hence by (4.2.4)

$$\varphi(x) = \varphi(K_j; x) = \varphi(K_j \cup K; x) = \varphi(K; x) = \varphi_0(x),$$

i.e. φ is an extension of φ_0.

Now we pass to the general linear equation (4.5.1), assuming thus $h \neq 0$.

Theorem 4.5.2. *Let hypotheses* (i)–(iii) *be fulfilled with f, g, h of form* (4.5.2) *and $q \leqq 0$. The set Φ of meromorphic solutions of equation* (4.5.1) *is determined as follows.*

(1) *If either $q < 0$ or* (4.5.7) *holds, then Φ consists of the single φ which is of form* (4.5.3) *with $k = r - q$ and Φ regular at the origin.*
(2) *If* (4.5.6) *holds with an $m < 0$, then $\Phi = \varnothing$ when $r = m$, and when $r > m$ Φ forms a one-parameter family whose members φ satisfy* (4.5.3) *with $k = m$ and Φs regular at the origin except for a unique φ_0 of the form*

$$\varphi_0(x) = x^r \Phi_0(x), \qquad \Phi_0(0) \neq 0,$$

with Φ_0 regular at the origin.

Proof. Inserting (4.5.2) and (4.5.3) into (4.5.1) we get

$$x^k(F(x))^k \Phi(f(x)) = x^{k+q} G(x)\Phi(x) + x^r H(x). \tag{4.5.8}$$

(1) If $q = 0$ and (4.5.7) holds then we must have $k = r = r - q$, and Φ can be uniquely determined from (4.5.8) in virtue of Theorem 4.2.3. If $q < 0$, then necessarily $k = r - q$ and Theorem 4.2.3 again works for equation (4.5.8), but now in a neighbourhood of the origin. The unique Φ resulting from Theorem 4.2.3 can be then uniquely extended onto X with the aid of Theorem 4.2.2.

(2) If (4.5.6) holds with an $m < 0$, then obviously $q = 0$, and (4.5.8) can be written as

$$x^k[(F(x))^k \Phi(f(x)) - G(x)\Phi(x)] = x^r H(x). \tag{4.5.9}$$

If $r = m$ and $k \neq m$, then the left-hand side of (4.5.9) has at the origin a pole of order different from r, a contradiction. For $k = m = r$ (4.5.9) yields $0 \neq H(0)/\Phi(0) = F(0)^m - G(0) = [f'(0)]^m - g(0) = 0$ which is also impossible. Thus if $r = m$ equation (4.5.1) has no solutions of the form (4.5.3).

If $r < m$, we take $k = r$ in (4.5.9) and determine a unique $\Phi = \Phi_0$ on account of Theorems 4.2.3 and 4.2.2. The theorem results now from Theorem 4.5.1. ■

Remark 4.5.1. If in (4.5.6) we have $m \geqq 0$ or $r < m < 0$, Theorem 4.5.2 does not work. Now $q = 0$ in (4.5.2) and $k = r$ in (4.5.3), so that a meromorphic φ satisfies (4.5.1) if and only if Φ is a local analytic solution of the equation

$$\Phi(x) = \frac{(F(x))^r}{G(x)} \Phi(f(x)) - \frac{H(x)}{G(x)}, \tag{4.5.10}$$

resulting from (4.5.8) on taking $k = r$. Equation (4.5.10), in turn, has such solutions if and only if it has formal ones, i.e. if system (4.2.6) associated with equation (4.5.10) has a solution (c_i).

We learn from Theorem 4.4.4 that we cannot handle the case of $q > 0$. Equation (4.5.1) may then have divergent formal solutions.

Comments. Meromorphic solutions of linear and nonlinear functional equations were investigated also by R. Raclis [1], W. Pranger [1], R. Goldstein [1]–[6].

4.6 Special equations

Some of the results of Sections 4.2–4.4 will be used here for finding local analytic solutions of the Schröder and Abel equations. We preserve the abbreviation 'LAS' from Section 4.2.

4.6A. The Schröder equation

Theorem 4.2.3 when applied to the equation

$$\sigma(f(x)) = s\sigma(x) \tag{4.6.1}$$

yields the famous theorem of G. Koenigs [1].

Theorem 4.6.1. *Let $X \subset \mathbb{C}$ be a neighbourhood of the origin, and let $f: X \to \mathbb{C}$ be an analytic function,*

$$f(0) = 0, \qquad f'(0) = s, \qquad 0 < |s| < 1.$$

Then equation (4.6.1) *has a unique LAS σ fulfilling the condition $\sigma'(0) = 1$. This solution is given by the formula*

$$\sigma(x) = \lim_{n\to\infty} s^{-n} f^n(x). \tag{4.6.2}$$

Proof. Because $s(f'(0))^k = s^{k+1} \neq 1$ for $k \in \mathbb{N}$, system (4.2.7), when written for equation (4.6.1), is uniquely solvable. Thus the existence of a unique LAS σ of (4.6.1) actually follows from Theorem 4.2.3. Formula (4.6.2) may be obtained by the argument we have used in the proof of Theorem 3.5.1 (formula (3.5.2)). ■

Remark 4.6.1. If an invertible function σ satisfies equation (4.6.1), then its inverse $\varphi = \sigma^{-1}$ satisfies the *Poincaré equation* (Poincaré [1], [2])

$$\varphi(sx) = f(\varphi(x)).$$

The case where $|s| = 1$, but s is not a root of unity, is covered by Theorem 4.3.1. For the case where s is a root of unity we have the following result (see Rausenberger [1], Muckenhoupt [1]).

Theorem 4.6.2. *Let the conditions of Theorem 4.6.1. be fulfilled except that now s is a pth root of unity. Then equation* (4.6.1) *has an LAS σ such that $\sigma(0) = 0$, $\sigma'(0) = 1$ if and only if $f^p = \mathrm{id}$. The solution is not unique.*

8.5A. Julia's equation and the iterative logarithm

The theory of conjugate FPSs is based on some notions and results by J. Ecalle [2] concerning the Julia equation

$$\lambda(f(x)) = f'(x)\lambda(x). \tag{8.5.1}$$

Let f be an FPS of the form

$$f(x) = x + \sum_{n=m}^{\infty} b_n x^n, \qquad b_m \neq 0,\ m \geqq 2. \tag{8.5.2}$$

Suppose that an FPS

$$\lambda(x) = \sum_{n=0}^{\infty} c_n x^n \tag{8.5.3}$$

formally satisfies equation (8.5.1). Insert (8.5.2) and (8.5.3) into (8.5.1) to get

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\binom{n}{i}c_n x^{n-i}\left(\sum_{k=m}^{\infty} b_k x^k\right)^i = \sum_{n=0}^{\infty}\sum_{i=0}^{n}(i+m)b_{i+m}c_{n-i}x^{n+m-1}.$$

Equating the coefficients of x^{m+j-1}, $j \in \mathbb{N}_0$, we obtain

$$jc_j b_m + A_j = mb_m c_j + B_j, \qquad A_0 = B_0 = 0, \tag{8.5.4}$$

where the terms A_j and B_j contain only c_i with $i < j$, and are homogeneous in the c_i. For $j = 0, \ldots, m-1$ relation (8.5.4) yields $c_j = 0$. Thus $A_m = B_m = 0$, c_m may be arbitrary, and then the c_j for $j > m$ can be uniquely determined from (8.5.4) and are homogeneous functions of c_m. Thus we have the following.

Theorem 8.5.1. *If f is an FPS of the form* (8.5.2), *then equation* (8.5.1) *has a unique formal solution λ_0 of the form*

$$\lambda_0(x) = b_m x^m + \sum_{n=m+1}^{\infty} c_n x^n. \tag{8.5.5}$$

The general formal solution λ of (8.5.1) *is given by $\lambda(x) = c\lambda_0(x)$, where c is an arbitrary constant (a parameter).*

Theorem 8.5.1 gives rise to the following definitions which are due to J. Ecalle [2].

Definition 8.5.1. (1) The FPS λ_0 given by (8.5.5) is said to be the *iterative logarithm* (logit f) of the FPS f given by (8.5.2) and is denoted by f_*.

(2) The number $m-1$ (see (8.5.5)) is called the *iterative valuation* (valit f) of f.

(3) By the *iterative residuum* (resit f) of f we mean the coefficient of x^{-1} in the formal Laurent series $1/f_*$, or, which is the same, the coefficient of x^{m-1} in the FPS $1/f_0$, where $f_*(x) = x^m f_0(x)$.

There is a one-to-one correspondence between f and logit f.

Theorem 8.5.2. *To every FPS* (8.5.5) *there corresponds a unique FPS* (8.5.2) *such that $\lambda_0 = \text{logit} f$.*

Proof. This follows on inserting (8.5.2) and (8.5.5) into (8.5.1) and equating the coefficients of x^{2m+j-1}. ∎

The convergence of the series (8.5.5), in the case where the series (8.5.2) converges in a neighbourhood of the origin, is a difficult problem. What is more, it is rather a rare situation, as may be seen from the theorem below whose proof will not be given here.

Theorem 8.5.3. *Let f be a meromorphic function, regular at the origin and having the expansion* (8.5.2). *If the FPS f_* has a positive radius of convergence, then*

$$f(x) = \frac{x}{1+bx}, \qquad b \in \mathbb{C}.$$

Theorem 8.5.3 is implied by the results of I. N. Baker [9] (see also Szekeres [4]) and Erdős–Jabotinsky [1].

8.5B. Formally conjugate power series

We start with necessary conditions for two FPSs

$$\left.\begin{aligned} f(x) &= x + \sum_{n=m}^{\infty} b_n x^n, \qquad b_m \neq 0, m \geqq 2,\\ g(x) &= x + \sum_{n=k}^{\infty} a_n x^n, \qquad a_k \neq 0, k \geqq 2, \end{aligned}\right\} \tag{8.5.6}$$

to be conjugate. Regarding the theorem that follows see Ecalle [1], [2], Erdős–Jabotinsky [1], Muckenhoupt [1].

Theorem 8.5.4. *Let f and g, of form* (8.5.6), *be formally conjugate FPSs, i.e.*

$$\varphi(f(x)) = g(\varphi(x)) \tag{8.5.7}$$

holds with an invertible FPS φ, the inverse of which has the form

$$\varphi^{-1}(x) = \sum_{n=1}^{\infty} d_n x^n, \qquad d_1 \neq 0. \tag{8.5.8}$$

Then, for $f_* = \text{logit} f$ and $g_* = \text{logit}\, g$, we have

$$f_* = (g_* \circ \varphi)/\varphi', \tag{8.5.9}$$

$$\text{valit} f = \text{valit}\, g, \tag{8.5.10}$$

$$\text{resit} f = \text{resit}\, g. \tag{8.5.11}$$

Proof. In view of Theorem 8.5.1 the series f_* and g_* are uniquely determined.

Inserting (8.5.6) and (8.5.8) into the relation $\varphi^{-1} \circ g = f \circ \varphi^{-1}$ we obtain

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \binom{n}{i} d_n x^{n-i} \left(\sum_{j=k}^{\infty} a_j x^j \right)^i = \sum_{n=m}^{\infty} b_n \left(\sum_{i=1}^{\infty} d_i x^i \right)^n$$

whence it follows immediately that $m = k$, yielding (8.5.10); and also $b_m = a_m d_1^{1-m}$. Hence the series on the right-hand side of (8.5.9) has the form $b_m x^m + \cdots$. Thus it is logit f, if it satisfies the Julia equation (8.5.1). We check that (note that $g_* \circ g = g' \circ g_*$, by the definition of g_*)

$$\frac{g_* \circ (\varphi \circ f)}{\varphi' \circ f} = \frac{g_* \circ (g \circ \varphi)}{\varphi' \circ f} = \frac{(g_* \circ g) \circ \varphi}{\varphi' \circ f} = \frac{g' \circ \varphi}{\varphi' \circ f} g_* \circ \varphi = \frac{(g' \circ \varphi)\varphi'}{\varphi' \circ f} \frac{g_* \circ \varphi}{\varphi'}.$$

When (8.5.7) is differentiated, the first ratio here becomes f', i.e.

$$\frac{g_* \circ \varphi}{\varphi'} \circ f = f' \cdot \frac{g_* \circ \varphi}{\varphi'}$$

and we get (8.5.9) in view of Theorem 8.5.1.

In order to calculate the iterative residua of g and f we need to find the coefficient of x^{-1} in $1/g_*$ and $1/f_*$, respectively. This coefficient depends only on a finite number of the coefficients in the FPSs g_* and φ (respectively in the FPSs f_* and φ). Thus there exist polynomials G and Φ (curtailments of g_* and φ) such that the coefficients of x^{-1} in $1/G$ and in $\Phi'/(G \circ \Phi)$ are equal to those in $1/g_*$ and in $1/f_* = \varphi'/(g_* \circ \varphi)$, respectively. But for polynomials these coefficients are the usual residua of the corresponding functions, and can be expressed by the known integral formulae. Thus we have

$$\operatorname{resit} f = \frac{1}{2\pi i} \int \frac{\Phi'(x)\mathrm{d}x}{G(\Phi(x))} = \frac{1}{2\pi i} \int \frac{\mathrm{d}u}{G(u)} = \operatorname{resit} g,$$

where the path of integration is a (positively oriented) closed contour around the origin. This proves (8.5.11). ■

The relation of conjugacy is transitive. Thus we may first look for conditions for f to be conjugate with a specific (sample) FPS. We attempt to find a $D \in \mathbb{C}$ for f to be formally conjugate with

$$g(x) = x + x^m + Dx^{2m-1}. \tag{8.5.12}$$

Supposing that (8.5.7) holds with an FPS φ, the inverse of which has expansion (8.5.8), we obtain

$$\sum_{n=m}^{\infty} b_n \left(\sum_{k=1}^{\infty} d_k x^k \right)^n = \sum_{p=1}^{\infty} \sum_{i=1}^{p} d_p \binom{p}{i} x^{p-i+mi}(1 + Dx^{m-1})^i. \tag{8.5.13}$$

Comparing in (8.5.13) the coefficients of x^m yields $b_m d_1^m = d_1$, whence d_1 may be any of the $m-1$ values of the $(m-1)$st root of b_m. Equating then the coefficients of x^{m+j-1} for $j \geqq 2$ we find

$$mb_m d_1^{m-1} d_j + A_j = jd_j + B_j, \tag{8.5.14}$$

where A_j and B_j depend only on d_i with $i < j$, and, for $j \geqq m$, B_j may depend also on D. Since $b_m d_1^{m-1} = 1$, relation (8.5.14) yields unique d_j for $j = 2, \ldots, m-1$. For $j = m$ we have $B_m = B_m^* + d_1 D$, where B_m^* is independent of D. Since (8.5.14) now becomes $A_m = B_m$, we get $D = d_1^{-1}(A_m - B_m^*)$. For $j > m$ we can again determine the d_j from (8.5.14), and d_m is arbitrary.

To proceed further we need the following lemma, which results from comparing the coefficients in the formula that we get from (8.5.1) with $\lambda = f_*$ (see (8.5.4)).

Lemma 8.5.1. *If*

$$f(x) = x + x^m + dx^{2m-1} + \sum_{n=2m}^{\infty} b_n x^n$$

then

$$f_*(x) = x^m + (D - \tfrac{1}{2}m)x^{2m-1} + \sum_{n=2m+2}^{\infty} c_n x^n.$$

Now, we can prove that D does not depend on the choice of the root d_1. For suppose that f given by (8.5.2) is formally conjugate with g defined by (8.5.12). By Lemma 8.5.1 $g_*(x) = x^m + (D - \frac{1}{2}m)x^{2m-1} + \cdots$, i.e. $(1/g_*)(x) = x^{-m} - (D - \frac{1}{2}m)x^{-1} + \cdots$, Theorem 8.5.4 says that $\operatorname{resit} f = \operatorname{resit} g = -D + \frac{1}{2}m$, and $D = \frac{1}{2}m - \operatorname{resit} f$ depends only on f.

It follows from what has been said so far that f is conjugate with g of form (8.5.12) if and only if $D = \frac{1}{2}m - \operatorname{resit} f$. Thus two series of form (8.5.6) are conjugate with the same g given by (8.5.12) if they have the same iterative valuation and iterative residuum. These considerations, together with Theorem 8.5.4, imply the following.

Theorem 8.5.5. *Let f and g be FPSs with $f(0) = g(0) = 0$, $f'(0) = g'(0) = 1$. Series f and g are formally conjugate if and only if* valit f = valit g *and* resit f = resit g.

There is one more fact that can be derived from our discussion based on examining relation (8.5.13). Namely, if we stop after having determined from (8.5.14) the d_j for $j = 1, \ldots, m-1$, then we get a polynomial $P(x) = d_1 x + \cdots + d_{m-1}x^{m-1}$ such that $\bar{g} := P^{-1} \circ f \circ P$ has a form $\bar{g}(x) = x + x^m + dx^{2m-1} +$ terms of higher orders, since the latter bear only on c_j with $j > m$. We shall formulate this fact as a lemma, which will be useful in the next subsection.

Lemma 8.5.2. *If f is an FPS of form* (8.5.2), *then there exists a polynomial P such that*

$$P^{-1} \circ f \circ P(x) = x + x^m + dx^{2m-1} + \cdots,$$

where $d = \frac{1}{2}m - \operatorname{resit} f$.

8.5C. Conjugate analytic functions

The conditions expressed in Theorem 8.5.5 are also necessary, but not sufficient for the analytic conjugacy of f and g if they actually represent analytic functions. In fact, let f be an analytic function with expansion (8.5.2) such that f_* has a positive radius of convergence, and let g be given by (8.5.12), where $D = \frac{1}{2}m - \operatorname{resit} f$. Then f and g are formally conjugate in virtue of Theorem 8.5.5. If f and g were analytically conjugate, then we would have $g = \varphi^{-1} \circ f \circ g$ with an analytic function φ, and by Theorem 8.5.4 $g_* = (f_* \circ \varphi)/\varphi'$ would have a positive radius of convergence. But this is impossible in view of Theorem 8.5.3, as $g(x) \neq x/(1+bx)$.

Necessary and sufficient conditions for analytic conjugacy of functions f, g with $f(0) = g(0) = 0$, $f'(0) = g'(0) = 1$ have been given by J. Ecalle [3], but they are too complicated to be reproduced here. However, after B. Muckenhoupt [1], we shall prove the following.

Theorem 8.5.6. *Let $X \subset \mathbb{C}$ be a neighbourhood of the origin, and let $f: X \to \mathbb{C}$ and $g: X \to \mathbb{C}$ be analytic functions such that $f(0) = g(0) = 0$, $f'(0) = g'(0) = 1$. If their iterative logarithms f_* and g_* have positive radii of convergence and f, g are formally conjugate, then they are analytically conjugate.*

Proof. By Lemma 8.5.2 and Theorem 8.5.5 it suffices to consider f and g of the form

$$f(x) = x + x^m + dx^{2m-1} + \sum_{n=2m}^{\infty} b_n x^n, \quad g(x) = x + x^m + dx^{2m-1} + \sum_{n=2m}^{\infty} a_n x^n,$$

where $d = \frac{1}{2}m + r$, $m = 1 + \operatorname{valit} f = 1 + \operatorname{valit} g$, $r = \operatorname{resit} f = \operatorname{resit} g$, for the series above are formally conjugate with the corresponding original ones.

By Lemma 8.5.1 we have

$$f_*(x) = x^m + (d - \tfrac{1}{2}m)x^{2m-1} + \cdots, \qquad g_*(x) = x^m + (d - \tfrac{1}{2}m)x^{2m-1} + \cdots. \tag{8.5.15}$$

Consider the differential equation

$$y' = f_*(y)/g_*(x). \tag{8.5.16}$$

We want to find its solutions of the form $y(x) = x + x^m z(x)$. Then z should satisfy

$$z' = \frac{f_*(x + x^m z) - g_*(x) - mzx^{m-1} g_*(x)}{x^m g_*(x)}. \tag{8.5.17}$$

By (8.5.15) the right-hand side of (8.5.17) is an analytic function of (x, z) in a neighbourhood of $(0, 0)$. By Cauchy's Existence Theorem for differential equations, (8.5.17) has an analytic solution z in a neighbourhood of the origin. The corresponding function y is analytic and invertible in a neighbourhood of the origin. Write

$$h = y^{-1} \circ f \circ y. \tag{8.5.18}$$

Treating h as an FPS we find similarly as in the first lines of the proof of Theorem 8.5.4 that

$$h(x) = x + x^m + \sum_{n=m+1}^{\infty} h_n x^n$$

(observe that now $b_m = 1$). By Theorem 8.5.1 the FPS $h_* = x^m + \cdots$ makes sense. We want to prove that $h_* = g_*$.

Indeed, (8.5.16), (8.5.18), the relation $f_* \circ f = f' \cdot f_*$ and again (8.5.16) imply

$$(y' \circ h)(g_* \circ h) = f_* \circ (y \circ h) = (f_* \circ f) \circ y = (f' \circ y)(f_* \circ y) = (f' \circ y)y' g_*.$$

But (8.5.18) yields $(y' \circ h)h' = (f' \circ y)y'$ so that

$$g_*(h(x)) = h'(x) g_*(x).$$

In view of (8.5.15) this means that $g_* = h_*$. By Theorem 8.5.2 also $g = h$ and (8.5.18) shows that g is analytically conjugate with f. ■

Now we pass to the case where $|f'(0)| = 1$ but $f'(0) \neq 1$. Observe that the condition $f'(0) = g'(0)$ is necessary for functions f and g (both analytic in a neighbourhood of the origin) to be analytically conjugate. Theorem 8.5.7, whose part (b) is due to B. Muckenhoupt [1] contains conditions equivalent to analytical conjugacy. (If $f'(0)^p = 1$, $f^p \neq \mathrm{id}$, then Theorem 8.7.6 applies.)

Theorem 8.5.7. *Let $X \subset \mathbb{C}$ be a neighbourhood of the origin and let $f: X \to \mathbb{C}$, $g: X \to \mathbb{C}$ be analytic functions such that $f(0) = g(0) = 0$. Consider two cases:*

(a) *$f'(0) \in S$, where S is the Siegel set (see Definition 4.3.1),*
(b) *$f'(0) \neq 1$ is a pth root of unity and $f^p = \mathrm{id}$.*

Necessary and sufficient conditions for g to be analytically conjugate with f are

in case (a): $g'(0) = f'(0)$,
in case (b): $g'(0) = f'(0)$ *and* $g^p = \mathrm{id}$.

Proof. Part (a) follows from Theorem 8.3.1. In case (b) the necessity of the condition $g^p = \mathrm{id}$ is obvious, and the sufficiency is a consequence of Theorem 4.6.2. ■

8.5D. Abel's equation

Results of Subsection 8.5A may also be used to prove a theorem on complex solutions to the Abel equation

$$\alpha(f(x)) = \alpha(x) + 1 \tag{8.5.19}$$

in the case where $s = f'(0) = 1$ (Ecalle [2]; for $s \neq 1$ see Subsection 4.6B).

Theorem 8.5.8. *Let f having expansion (8.5.2) be analytic in a neighbourhood* *of the origin. Then equation (8.5.19) has solutions α defined in a vicinity of the origin and such that*

$$\alpha(x)=c_0\log x+\sum_{i=1}^{m-1}c_ix^{-i}+\varphi(x),\qquad c_0,\ldots,c_{m-1}\in\mathbb{C},\quad (8.5.20)$$

where φ is an analytic function in a neighbourhood of the origin, if and only if $f_*=\operatorname{logit}f$ has a positive radius of convergence. If f_* actually is a function analytic at the origin, then α is determined uniquely up to an additive constant and is given by the formula

$$\alpha(x)=c+\int_{x_0}^{x}\left[\int_{x_0}^{f(x_0)}\frac{dt}{f_*(t)}\right]^{-1}\frac{dt}{f_*(t)},\quad (8.5.21)$$

where c is an arbitrary constant (a parameter), x_0 is a point arbitrarily fixed in a vicinity V of the origin, and the integration is over an arbitrary path in V joining x_0 with x, respectively $f(x_0)$.

Proof. Let α, defined in a vicinity of the origin, be a solution of (8.5.19) with property (8.5.20). Since (8.5.19) cannot have a solution analytic at the origin, not all the c_i are zero. Thus $\lambda(x)=1/\alpha'(x)$ is analytic at the origin and satisfies equation (8.5.1). In view of Theorem 8.5.1, we have $\lambda(x)=\gamma f_*(x)$ with a $\gamma\neq 0$. Since $f_*(x)=x^m\psi(x)$ with $\psi(0)=b_m\neq 0$, the function $1/\lambda$ is analytic in a vicinity V of the origin and has a pole of order m at 0. Consequently, in V we have

$$\alpha(x)=c+\int_{x_0}^{x}\frac{dt}{\lambda(t)}=c+\gamma^{-1}\int_{x_0}^{x}\frac{dt}{f_*(t)},\quad (8.5.22)$$

where $x_0\in V$ is arbitrary, and we integrate over an arbitrary path in V joining x_0 with x. Since α satisfies (8.5.19), we have

$$\int_{x_0}^{f(x_0)}\frac{dt}{f_*(t)}=\int_{x_0}^{f(x_0)}\alpha'(t)dt=\gamma[a(f(x_0))-\alpha(x_0)]=\gamma,$$

and so (8.5.21) results from (8.5.22).

Now assume that f_* is an analytic function in a neighbourhood of the origin. Then in a vicinity of $x=0$ we have $f_*(x)\neq 0$, $f(x)\neq -x$ and

$$\lim_{x\to 0}\int_{x}^{f(x)}\frac{dt}{f_*(t)}=\lim_{x\to 0}\frac{f(x)^{1-m}-x^{1-m}}{(1-m)b_m}=1,\quad (8.5.23)$$

where the integral is taken over the segment joining x and $f(x)$. So we can find a vicinity V of the origin such that both f_* and the integral occurring in (8.5.23) do not vanish in V. Define $\alpha(x)$ for $x\in V$ by (8.5.21) with $x_0\in V$ arbitrarily fixed. Then it is easily seen that α has property (8.5.20) and satisfies equation (8.5.19). ■

Comments. Regarding conjugacy problems we discussed in this and the preceding sections see, in particular, Belickiĭ [1]–[9], Bratman [1], Ditor [1], Ecalle [1]–[3], Fine–Kostant [1], Herman [1], Julia [3], Kneser [1], Kuczma [11], [23], Muckenhoupt [1], Nitecki [1], Sternberg [2], [6], Venti [1], Weitkämper [2].

8.6 Permutable functions

Let $g=f$ in the conjugacy equation (8.0.1). Then we obtain the equation

$$\varphi(f(x))=f(\varphi(x)).\quad (8.6.1)$$

If (8.6.1) is satisfied, then we say that f and φ *commute* or are *permutable*.

Commuting functions have been extensively studied by many authors (see Comments at the end of this section). Here we confine ourselves to the one-dimensional case only and to presenting some results which can be obtained with the aid of solutions of the Schröder, Abel and Böttcher equations.

Permutability is a rare property of functions. In the Cartesian square of the space $\mathbf{C}=C([0,1],\mathbb{R})$ of continuous functions with the usual sup norm, the pairs of commuting functions form a nowhere dense set (Kuczma [27]). On the other hand, it can be deduced from Theorem 5.3.1 that for every strictly monotonic $f\in\mathbf{C}$ the solution $\varphi\in\mathbf{C}$ of equation (8.6.1) depends on an arbitrary function (Lipiński [1], Kuczma [26, pp. 213–14]).

However, if f and φ have a higher regularity, then often we are able to prove the uniqueness of φ satisfying (8.6.1) with a given f. The theorems to this effect we present below are based on the following scheme.

Suppose that f is conjugate to a function g, $\varphi_0\circ f=g\circ\varphi_0$, where φ_0 is an invertible function smooth enough. If a function φ is permutable with f, then $\phi:=\varphi_0\circ\varphi$ satisfies equation (8.0.1). In a class of sufficiently smooth functions this equation may happen to have a unique solution, up to a parameter. In such a case, since both ϕ and φ_0 satisfy equation (8.0.1), they must be related in a form $\phi=G(c,\varphi_0)$, where c is the parameter. Hence $\varphi=\varphi_0^{-1}\circ G(c,\varphi_0)$, and we obtain a one-parameter family of solutions of (8.6.1).

Now, results on smooth permutable functions are obtained by the argument just described with $g(x)=sx$, $g(x)=x+1$ and $g(x)=x^p$. Equation (8.0.1) then becomes the equation of Schröder (8.0.2), Abel (8.0.3) and Böttcher (8.0.4), respectively.

In the following, when we say 'all φ' we mean '*all functions φ commuting with f and satisfying $\varphi(0)=0$*'. Moreover, whenever we write $U\subset X$, we mean by U a neighbourhood of the origin. Finally, we put

$$s:=f'(0).$$

In the case of analytic functions, if $0<|s|\leqslant 1$, we obtain from Theorems 4.3.1, 4.6.1 and 4.6.3 the following.

Following G. Szekeres [1] we distinguish special solutions of this kind which are called the principal solutions to Schröder's and Abel's equations.

In the sequel we take $X=[0,a|$, $0<a\leqslant\infty$, and $f\colon X\to X$ to be a continuous strictly increasing function, $0<f(x)<x$ in $X\setminus\{0\}$.

Definition 9.1.1. Let us assume that the limit

$$s:=\lim_{n\to\infty}\frac{f^{n+1}(x)}{f^n(x)},\qquad x\in X\setminus\{0\},$$

exists and it does not depend on x. Take an $x_0\in X\setminus\{0\}$. If the limit

$$\sigma(x)=\lim_{n\to\infty}\frac{f^n(x)}{f^n(x_0)},\qquad x\in X,\tag{9.1.1}$$

exists and is positive and finite in $X\setminus\{0\}$, then it satisfies

$$\sigma(f(x))=s\sigma(x)\tag{9.1.2}$$

and is called the *principal solution of the Schröder equation* (9.1.2).

Table 9.1. *Schröder's equation*

Function class	Subsection	Applications	Section, etc.
Monotonic	2.3F	Branching processes (BP): limit distributions	2.1B, 2.6A
Convex	2.4D	BP: restricted stationary measures	2.6C
		Fractional iteration	11.4
Regularly varying	2.5B	BP: norming sequences	Note 2.8.14
C^r, $r\geqslant 1$	3.5A	Linearization	8.2
	8.2B	Permutability	8.6
		Fractional iteration	11.4
		Cross ratio	3.5D
		An integral equation	Note 3.8.14
Analytic	4.6A	Conjugacy	8.4A
	4.3A	Permutability	8.6
	8.2A	Fractional iteration	11.6
		Characterization of functions	10.2A, D
		Second-order iterative inequality	12.7

Table 9.2. *Abel's equation*

Function class	Subsection	Applications	Section, etc.
Convex	2.4D	BP: limit distributions	2.1B, 2.6A
		BP: restricted stationary measures	2.1D, 2.6C
C^r, $r\geqslant 1$	3.5C	Second-order differential equations	3.7C
		Conjugacy	8.4B
		Permutability	8.6
Analytic	4.6B		
	8.5D		

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we took in (9.1.1) an $x_1\in X\setminus\{0\}$ in place of x_0, then the limit would from the previous one only by a constant multiple. Thus the principal tion of the Schröder equation is unique up to a constant factor.

general, the principal solution behaves better near the origin than other ions of (9.1.2). Conditions for the existence of limit (9.1.1) are contained heorems 2.3.12 and 2.4.4 and also in Lemma 2.5.1.

rk 9.1.1. If there exists the limit

$$\tilde\sigma(x)=\lim_{n\to\infty}s^{-n}f^n(x),\qquad x\in X,\tag{9.1.3}$$

limit (9.1.1) exists as well. The converse is not true. E. Seneta [5] proved the key condition for the existence of $\tilde\sigma$, $0<\tilde\sigma<\infty$, is the convergence improper integral

$$\int_0^\delta (f(x)-sx)x^{-2}\,dx$$

$X\setminus\{0\}$ (see Theorem 1.3.2). Formula (9.1.3) is called the *Koenigs thm* (Koenigs [1]); see Theorem 3.5.1.

w let $(d_n)_{n\in\mathbb N}$ be any sequence of reals for which

$$\lim_{n\to\infty}\frac{1}{d_n}(f^{n+1}(x)-f^n(x))=1,\qquad x\in X\setminus\{0\}.\tag{9.1.4}$$

tion 9.1.2. Let a sequence (d_n) have property (9.1.4). If there exists the

$$\alpha(x):=\lim_{n\to\infty}\frac{1}{d_n}(f^n(x)-f^n(x_0)),\qquad x\in X\setminus\{0\},\tag{9.1.5}$$

$x_0\in X\setminus\{0\}$, then it satisfies

$$\alpha(f(x))=\alpha(x)+1\tag{9.1.6}$$

called the *principal solution of the Abel equation* (9.1.6).

may be easily shown that limit (9.1.5) does not depend on the choice sequence (d_n) satisfying (9.1.4), and that by replacing x_0 in (9.1.5) by her $x_1\in X\setminus\{0\}$ we obtain a limit which differs from (9.1.5) by a constant mand. Thus the principal solution of the Abel equation is determined an additive constant.

rk 9.1.2. If (9.1.4) holds for $d_n=f^{n+1}(x_0)-f^n(x_0)$, where $x_0\in X\setminus\{0\}$ f there exists the limit

$$\tilde\alpha(x)=\lim_{n\to\infty}\frac{f^n(x)-f^n(x_0)}{f^{n+1}(x_0)-f^n(x_0)},\qquad x\in X\setminus\{0\},\tag{9.1.7}$$

it is the principal solution of (9.1.6). Formula (9.1.7) is called the

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(This is a result of W. Sierpiński [1]. The proof is similar to that of Proposition 10.6.1, and is presented e.g. in Kuczma [26, Ch. XI, § 5].)

10.6.2. For results concerning other characterizations of peculiar, mainly c.n.d., functions, in most cases by systems of simultaneous equations see Andreoli [1], de Rham [1]–[5], Dubuc [3], Julia [4], Ruziewicz [1], Sierpiński [3], [4], Wunderlich [1]. Regarding systems of simultaneous iterative functional equations see also Howroyd [1]–[3].

10.6.3. The cotangent can be characterized in the following ways:

(a) (Gupta [1], private communication) The unique function $\varphi\colon (0, 1) \to \mathbb{R}$ that is continuous, satisfies $\lim_{x\to 0}(\varphi(x) - 1/x) = 0$ and solves the equations

$$\varphi(x) = \tfrac{1}{2}[\varphi(\tfrac{1}{2}x) + \varphi(\tfrac{1}{2}(x+1))], \qquad x \in (0, 1), \tag{10.6.2}$$

and $\varphi(x) + \varphi(1-x) = 0$ (thus $\varphi(\frac{1}{2}) = 0$) is $\varphi(x) = \pi \cot \pi x$.

(b) (Jäger [1]) If $\varphi\colon (0, 1) \to \mathbb{R}$ is continuous and satisfies the equations

$$\varphi(x) = \frac{1}{n}\sum_{i=0}^{n-1} \varphi\left(\frac{x+i}{n}\right), \qquad x \in (0, 1), \tag{10.6.3}$$

for all $n \in \mathbb{N}$, then $\varphi(x) = a \cot \pi x + b$, $a, b \in \mathbb{R}$.

The main tool to prove (a) is a particular case of Lemma 10.4.1 due to E. Mohr [1] (see also Walter [1]) that concerns Riemann integrable solutions (on [0, 1]) of (10.6.2). Using it for the (continuous in [0, 1]) solution $\tilde{\varphi}$ of (10.6.2), $\tilde{\varphi}(x) := \varphi(x) - \pi \cot \pi x$, $x \in (0, 1)$, $\tilde{\varphi}(0) = \tilde{\varphi}(1) = 0$, one finds $\tilde{\varphi} = 0$ by $\tilde{\varphi}(\frac{1}{2}) = \varphi(\frac{1}{2}) = 0$.

That the cotangent satisfies (10.6.3) for $n \in \mathbb{N}$ is seen, for instance, from its resolution into partial fractions

$$\pi \cot \pi x = \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{x+k} - \frac{1}{x-k}\right), \qquad x \notin \mathbb{Z}. \tag{10.6.4}$$

This formula is derived (Mohr [1], see also Walter [1]) as in (a), by showing that the function $\hat{\varphi}(x) := \pi \cot \pi x - \lim_{n\to\infty} \sum_{j=-n}^{n} 1/(x+j)$ for $x \in (0, 1)$, $\hat{\varphi}(0) = \hat{\varphi}(1) = 0$ shares all the properties of $\tilde{\varphi}$. In turn, (10.6.4) follows directly from Euler's identity for the sine,

$$\sin \pi x = \pi x \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right), \qquad x \in \mathbb{R}, \tag{10.6.5}$$

by differentiation. To prove (10.6.5) H. Haruki [5] uses the equation

$$\varphi(x) = \varphi(\tfrac{1}{2}x)\varphi(\tfrac{1}{2}(1-x)), \qquad x \in \mathbb{C}, \tag{10.6.6}$$

and shows that it has the unique entire solution $\varphi = 1$ and that the quotient of both sides of (10.6.5) is also an entire function (its singularities at the integers are removable) that satisfies (10.6.6).

11

Iterative roots and invariant curves

11.0 Introduction

Dealing with a continuous iteration semigroup $\{f^s\colon s > 0\}$ (see Section 1.7) and putting $f := f^1$ we get, in particular, the existence of solutions of the functional equations

$$\varphi^N = f \tag{11.0.1}$$

for any positive integer N; indeed, it suffices to take $\varphi := f^{1/N}$. Having, however, a self-mapping f which, *a priori*, is not embeddable into any continuous (or even only rational) iteration semigroup, we may go on trying to find solutions of (11.0.1), at least for a given $N \in \mathbb{N}$. This is just the central idea of the present chapter.

For obvious intuitive reasons any solution of equation (11.0.1) is called the *Nth iterative root* of the function f. The symbol $f^{1/N}$ used above suggests also an alternative term: (the Nth) *fractional iteration*, actually occurring in many papers.

The material of this chapter is organized as follows. Starting with the iterative roots of arbitrary mappings we proceed to discuss special aspects of this problem for functions from a subset of the real line (usually an interval) into itself. Next, we investigate local analytic solutions of equation (11.0.1) in a neighbourhood of the origin at the complex plane. Then we deal with iterative roots of identity, i.e. we seek for solutions of the so-called *Babbage equation*

$$\varphi^N = \mathrm{id} \tag{11.0.2}$$

which in fact is one of the oldest iterative functional equations ever discussed. We shall be concerned mainly with continuous solutions of (11.0.2) and of its special case $N = 2$ defining the *involutions*.

The last part of the chapter is loosely related with the preceding ones. The goal is to find two-dimensional manifolds invariant with respect to a given transformation. This interesting geometrical problem leads, in a particular case, to the following functional equation:

$$\varphi(f(x, \varphi(x))) = g(x, \varphi(x)). \tag{11.0.3}$$

The problem of existence and uniqueness of local Lipschitzian solutions of (11.0.3) in a neighbourhood of the origin is considered. The reason for associating such studies with finding iterative roots is just the following common feature of (11.0.1) and (11.0.3): both of them contain superpositions of the unknown function.

As previously, supplementary facts and information are collected in the final Notes section.

11.1 Purely set-theoretical case

In the whole of this short section, X denotes an arbitrary set, f is a self-mapping of X and N is a fixed positive integer.

We start with the following.

Theorem 11.1.1. *Let $\varphi: X \to X$ be a solution of the equation $\varphi^N = f$. Then φ is surjective (resp. injective, bijective) if and only if f is surjective (resp. injective, bijective).*

Proof. $\varphi(X) = X$ implies $f(X) = \varphi^N(X) = X$. Conversely, assume f to be surjective; if the set $X \setminus \varphi(X)$ were not empty, say $x_0 \in X \setminus \varphi(X)$ then, for any $x \in X$, we would have $\varphi(x) \neq x_0$ whence $f(x) = \varphi^N(x) = \varphi(\varphi^{N-1}(x)) \neq x_0$, which contradicts the surjectivity of f.

Since the composition of injections is an injection again we see that the injectivity of φ implies that of f. Conversely, assume f is injective; if φ were not injective then, for some $x, y \in X, x \neq y$, we would have $\varphi(x) = \varphi(y)$ whence

$$f(x) = \varphi^N(x) = \varphi^{N-1}(\varphi(x)) = \varphi^{N-1}(\varphi(y)) = \varphi^N(y) = f(y),$$

which contradicts the injectivity of f.

The appropriate bijectivity equivalence follows now immediately. ■

For bijective mappings f the equation

$$\varphi^N = f \tag{11.1.1}$$

was solved by S. Łojasiewicz [1] (see also Bajraktarević [9], Haĭdukov [1] and Kuczma [26]). The general solution in the case $N = 2$ has been described by R. Isaacs [1]. Extending the notions and methods of Isaacs (see also Sklar [1]), G. Zimmermann-Riggert [1], [2] gave a solution in the general case by reducing the problem of the existence of the Nth iterative roots to the problem of the so-called *N-mateability*. The whole procedure is described in detail in G. Targoński's monograph [8]. We shall confine ourselves here to the statement of the main result to give a flavour of such studies only.

For let n be a divisor of $N \geqq 2$ and let $\Omega_1, \ldots, \Omega_n$ be orbits of f. Put $\Omega := \bigcup_{i=1}^n \Omega_i$. We say that the orbits $\Omega_1, \ldots, \Omega_n$ are *N-mateable* provided there exists a map $\varphi_\Omega: \Omega \to \Omega$ such that $\varphi_\Omega^N = f|\Omega$ and φ_Ω has one single orbit.

Theorem 11.1.2. *Equation* (11.1.1) *has a solution if and only if the family of all orbits of F admits a decomposition into a disjoint union of classes such that the cardinality of each class is a divisor of N and the elements of each class are N-mateable.*

Various necessary and sufficient conditions for N-mateability may be found in Targoński's book [8], too.

For further purposes we shall end this section with the proof of the following lemma which is a particular case of a result of R. Isaacs [1].

Lemma 11.1.1. *Suppose that $f(a) = b$ and $f(b) = a$ for some $a, b \in X$, $a \neq b$. If, for any $x \in X$, the equality $f^2(x) = x$ implies that $x \in \{a, b, f(x)\}$, then the equation*

$$\varphi^2 = f \tag{11.1.2}$$

has no solutions.

Proof. Suppose that $\varphi: X \to X$ is a solution of (11.1.2) and put $c := \varphi(a)$. Then $f^2(c) = \varphi^5(a) = \varphi(f^2(a)) = \varphi(a) = c$ and therefore $c \in \{a, b, f(c)\}$. If $c = a$, then $b = f(a) = \varphi^2(a) = \varphi(c) = \varphi(a) = c = a$, a contradiction. If $c = b$, then $b = f(a) = \varphi^2(a) = \varphi(c) = \varphi(b)$ whence $a = f(b) = \varphi^2(b) = b$, a contradiction again. Finally, if we had $f(c) = c$, then $\varphi(b) = \varphi(f(a)) = \varphi^3(a) = f(\varphi(a)) = f(c) = c$ whence $\varphi(a) = \varphi(b)$ and, consequently, we would get $b = fa) = \varphi^2(a) = \varphi^2(b) = f(b) = a$, contrary to our assumption. ■

11.2 Continuous and monotonic solutions

Such a situation as described in Lemma 11.1.1 cannot happen for any increasing function $f: \mathbb{R} \to \mathbb{R}$. Actually, as we shall see later, any continuous and strictly increasing function on the real line possesses continuous and strictly increasing iterative roots of all orders. But first we shall prove some preliminary results.

Lemma 11.2.1. *Let $X \subset \mathbb{R}$ be an arbitrary set and let $f: X \to X$ be strictly monotonic. Assume $\varphi: X \to X$ to be a monotonic iterative root of f and fix a point $x_0 \in X$.*

(a) *If φ is increasing, then the following conditions are equivalent:*

(1) $f(x_0) = x_0$;

(2) $\varphi(x_0) = x_0$;

(3) $\varphi(x_0) = f(x_0)$.

(b) *If φ and f are decreasing, then $\varphi(x_0) = f(x_0)$ if and only if $f^2(x_0) = x_0$.*

Proof. Suppose that φ is an iterative root of the Nth order. By Theorem 11.1.1 φ is strictly monotonic.

To prove (a) assume (1) and suppose that $\varphi(x_0) > x_0$. Then $\varphi^{k+1}(x_0) > \varphi^k(x_0)$ for all $k \in \mathbb{N}$ whence $f(x_0) = \varphi^N(x_0) > \varphi(x_0) > x_0$, a contradiction. Similarly, $\varphi(x_0) < x_0$ would imply $f(x_0) < x_0$. Therefore, implication (1) $\Rightarrow$ (2) has been proved. Assume (2); then $f(x_0) = \varphi^N(x_0) = x_0 = \varphi(x_0)$, i.e. (3) holds true. Now, suppose (3) to be satisfied and put $g := \varphi^{N-1}$. Then g is strictly increasing and $g(x_0) = \varphi^{N-1}(x_0) = x_0$; applying the first part of the proof with f replaced by g we infer that $f(x_0) = \varphi(x_0) = x_0$.

To prove (b) observe that N has to be odd, $\psi := \varphi^2$, $g := f^2$ are strictly increasing and $\psi^N = g$. On account of (a), relation $g(x_0) = x_0$ is equivalent to the equality $\psi(x_0) = x_0$, i.e. $f^2(x_0) = x_0$ is equivalent to $\varphi^2(x_0) = x_0$. The latter equality is satisfied if and only if $\varphi^{N-1}(x_0) = x_0$ (recall that N is odd) being equivalent to $f(x_0) = \varphi(x_0)$. ■

Lemma 11.2.2. *Let $X \subset \mathbb{R}$ be an arbitrary set and let $f: X \to X$ be strictly increasing. Assume $\varphi: X \to X$ to be an increasing iterative root of f. Then the functions $(f - \mathrm{id}_X)(\varphi - \mathrm{id}_X)$ and $(\varphi - \mathrm{id}_X)(f - \varphi)$ are both nonnegative.*

Proof. Suppose that φ is an iterative root of the Nth order. By Theorem 11.1.1 φ is strictly increasing. Fix an $x_0 \in X$ and assume that $f(x_0) \geqslant x_0$. If we had $\varphi(x_0) < x_0$, then $\varphi^{k+1}(x_0) < \varphi^k(x_0)$ for all $k \in \mathbb{N}$ and we would get $f(x_0) = \varphi^N(x_0) < \varphi(x_0) < x_0$, a contradiction. Similarly, $f(x_0) \leqslant x_0$ implies $\varphi(x_0) \leqslant x_0$. This proves that the function $(f - \mathrm{id}_X)(\varphi - \mathrm{id}_X)$ is nonnegative. The remaining part of the proof is similar. ■

Theorem 11.2.1. *Let $X \subset \mathbb{R}$ be an interval and let $f: X \to X$ be a strictly monotonic surjection. Assume $\varphi: X \to X$ to be an iterative root of f. Then φ is continuous if and only if φ is strictly monotonic.*

Proof. By Theorem 11.1.1 φ is bijective. Continuous bijection on an interval has to be strictly monotonic. The converse results from the fact that any monotonic surjection on an interval is continuous. ■

11.2A. Strictly increasing continuous iterative roots

Now we can prove the following.

Theorem 11.2.2. *Let $X \subset \mathbb{R}$ be an interval and let f be a strictly increasing and continuous self-mapping of X. Then f possesses strictly increasing and continuous iterative roots of all orders. More precisely, for any positive integer $N \geqslant 2$, the strictly increasing and continuous solution of the functional equation*

$$\varphi^N(x) = f(x), \qquad x \in X,$$

depends on an arbitrary function.

Proof. Without loss of generality we may assume that $X = (a, b|$, $-\infty \leqslant a < b \leqslant \infty$, and $a < f(x) < x$ for all $x \in X$. Indeed, otherwise, setting $F := \{x \in X: f(x) = x\}$ we have $X = F \cup \bigcup_k X_k$ where X_k are pairwise disjoint intervals of the form $(\alpha, \beta|$ or $[\alpha, \beta)$ with $\alpha, \beta \in F$ or $\alpha = a$ or $\beta = b$. If φ is a (necessarily strictly) increasing iterative root of f, then Lemma 11.2.1 gives $\varphi(x) = x$ for $x \in F$, whence $\varphi(X_k) \subset X_k$ for each k; conversely, if for each k a self-mapping φ_k of X_k is a continuous Nth iterative root of $f|_{X_k}$, then the function $\varphi: X \to X$ defined by the formula

$$\varphi(x) := \begin{cases} \varphi_k(x) & \text{for } x \in X_k, \\ x & \text{for } x \in F, \end{cases}$$

yields a continuous and strictly increasing Nth iterative root of f (see Lemma 11.2.2).

Thus, to proceed, fix arbitrarily a point $x_0 \in X$, choose any points $x_1 > x_2 > \cdots > x_{N-1}$ from the interval $(f(x_0), x_0)$ and put

$$x_{k+N} := f(x_k)$$

(which is equivalent to $x_k = f^{-1}(x_{k+N})$) for all those $k \in \mathbb{Z}$ for which the recurrence procedure is performable. Put $J := \mathbb{Z} \cap [-k_0, \infty)$ provided that either $x_{-k_0} = b \in X$ or x_{-k_0+1} does exist but x_{-k_0} does not; otherwise, put $J := \mathbb{Z}$. Let $I_k := [x_{k+1}, x_k]$, $k \in \mathbb{Z}$, if $J = \mathbb{Z}$; if, however, $J = \mathbb{Z} \cap (-k_0, \infty)$, then

$$I_k := \begin{cases} [x_{k+1}, x_k] & \text{for } k \in \mathbb{Z} \cap (-k_0, \infty), \\ [x_{-k_0}, b] \cap X & \text{for } k = k_0. \end{cases}$$

Finally, given arbitrary increasing homeomorphisms φ_k of I_k onto I_{k+1}, $k \in \{0, \ldots, N-2\}$, we put

$$\varphi_k(x) := f \circ \varphi_{k-N+1}^{-1} \circ \cdots \circ \varphi_{k-1}^{-1}(x), \qquad x \in I_k,$$

for $k \in \mathbb{N}$, $k \geqq N-1$, and

$$\varphi_k(x) := \varphi_{k+1}^{-1} \circ \cdots \circ \varphi_{k+N+1}^{-1} \circ f(x), \qquad x \in I_k,$$

for $k \in y \cap (-\infty, -1]$. Now, it is easy to check that the formula

$$\varphi(x) := \varphi_k(x) \qquad \text{for } x \in I_k,\ k \in J,$$

defines a continuous and strictly increasing Nth iterative root of f. ■

Remark 11.2.1. Each continuous and increasing iterative root of a continuous and strictly increasing self-mapping of a real interval may be obtained in the manner just described. We omit the obvious detailed calculation.

11.2B. Strictly decreasing roots of strictly decreasing functions

If f is strictly decreasing, then, by Theorem 11.2.1, any continuous iterative root of f has to be strictly decreasing, too (because a composition of increasing functions remains increasing). Hence, the order of the root must be odd; there is no continuous even iterative root of a strictly decreasing function. Generally, the behaviour of iterative roots for decreasing functions gives less satisfaction than that for increasing ones.

What we are able to prove is the following.

Theorem 11.2.3. *Let X, Y be two disjoint real intervals and let Z be their union. Assume f to be a strictly decreasing and continuous self-mapping of Z such that*

$$f(X)=Y \qquad \text{and} \qquad f(Y)=X.$$

Then, for every odd $N\in\mathbb{N}$ and any strictly increasing solution $\psi\colon X\to X$ of the equation

$$\psi^N=f^2, \tag{11.2.1}$$

the formula

$$\varphi(x):=\begin{cases} f\circ\psi^{-\frac{1}{2}(N-1)}(x) & \text{for } x\in X,\\ \psi^{\frac{1}{2}(N+1)}\circ f^{-1}(x) & \text{for } x\in Y;\end{cases} \tag{11.2.2}$$

defines a continuous and strictly decreasing Nth iterative root of f. Conversely, each Nth iterative root of f has the above representation.

Proof. Without loss of generality we may assume that $N\geqslant 3$. Let $\psi\colon X\to X$ be an arbitrary continuous and (necessarily strictly) increasing solution of (11.2.1). Then the map $\varphi\colon Z\to Z$ given by formula (11.2.2) is continuous and strictly decreasing. Moreover, for any $x\in X$ one has

$$\varphi^2(x)=\psi^{\frac{1}{2}(N+1)}\circ f^{-1}\circ f\circ\psi^{-\frac{1}{2}(N-1)}(x)=\psi(x)$$

whereas, for any $x\in Y$, one gets

$$\varphi^2(x)=f\circ\psi^{-\frac{1}{2}(N+1)}\circ\psi^{\frac{1}{2}(N+1)}\circ f^{-1}(x)=f\circ\psi\circ f^{-1}(x).$$

Consequently, owing to the oddness of N, we obtain respectively

$$\varphi^N(x)=\varphi\circ\varphi^{N-1}(x)=\varphi\circ\psi^{\frac{1}{2}(N-1)}(x)=f\circ\psi^{-\frac{1}{2}(N-1)}\circ\psi^{\frac{1}{2}(N-1)}(x)=f(x),$$

$x\in X$, whereas

$$\begin{aligned}\varphi^N(x)&=\varphi\circ\varphi^{N-1}(x)=\varphi\circ f\circ\psi^{\frac{1}{2}(N-1)}\circ f^{-1}(x)\\ &=\psi^{\frac{1}{2}(N+1)}\circ f^{-1}\circ f\circ\psi^{\frac{1}{2}(N-1)}\circ f^{-1}(x)=\psi^N\circ f^{-1}(x)=f(x), \qquad x\in Y,\end{aligned}$$

on account of (11.2.1).

To prove the converse it suffices to take $\psi:=\varphi^2$. ■

Theorem 11.2.3 covers more cases than it might seem at the first glance, and its seemingly artificial assumptions are quite adequate. In fact, observe first that, according to Theorem 11.2.2, equation (11.2.1) has always a strictly increasing and (necessarily) continuous solution. Take any surjective and, strictly decreasing and continuous self-mapping f of an interval $X\subset\mathbb{R}$. Put $F:=\{x\in X\colon f^2(x)=x\}$; plainly, F is closed in X. Let intervals X_k, $k\in K\subset\mathbb{N}$, denote the components of $X\setminus F$. The function f has exactly one fixed point $x_0\in X$ which, obviously, belongs to F. Let K^- be the set of all $k\in K$ such

that X_k lies to the left of x_0. Put $Y_k:=f(X_k)$, $k\in K^-$. Clearly, $Y_k\cap X_k=\varnothing$ for $k\in K^-$. Moreover, $f(Y_k)=f^2(X_k)\setminus X_k$ since the endpoints of X_k are fixed points of f^2 (or, possibly, the left point of X_k coincides with the left point of X). Consequently, f maps the union $Z_k:=X_k\cup Y_k$, $k\in K^-$, onto itself. This reduces the situation to the case considered in Theorem 11.2.3. Therefore, for each $k\in K^-$, there exists a strictly decreasing and continuous solution φ_k of the equation $\varphi^N=f|_{Z_k}$. If x_k is an endpoint of X_k, then $\lim_{x\to x_k}\varphi_k^2(x)=x_k$ by means of Lemma 11.2.2 whence $\lim_{x\to x_k}\varphi_k(x)=\lim_{x\to x_k}f\circ\varphi_k^{1-N}(x)=f(x_k)$, $k\in K$. Thus, the function

$$\varphi(x):=\begin{cases}\varphi_k(x) & \text{for } x\in Z_k,\\ f(x) & \text{for } x\in F,\end{cases}$$

yields a strictly decreasing and continuous solution of the functional equation $\varphi^N(x)=f(x)$, $x\in X$. Consequently, we have proved the following.

Theorem 11.2.4. *Let $X\subset\mathbb{R}$ be an interval and let f be a strictly decreasing and continuous function from X onto X. For each odd $N\in\mathbb{N}$ there exists a strictly decreasing and continuous Nth iterative root of f.*

11.2C. Strictly decreasing roots of strictly increasing functions

If N is an even number, then a strictly increasing and continuous self-mapping f of a real interval X can have also continuous and strictly decreasing Nth iterative roots. In fact, the problem reduces itself to a solution of the system

$$\varphi^2=\psi \qquad \text{and} \qquad \psi^{\frac{1}{2}N}=f,$$

where ψ is increasing. In the light of Theorem 11.2.2, we may confine ourselves to the case $N=2$ of equation (11.0.1). We begin with

Theorem 11.2.5. *Let f be an increasing homeomorphism of an open or closed real interval X onto X. Assume that $\xi\in F:=\{x\in X\colon f(x)=x\}$ and put $F^-:=\{x\in F\colon x\leqslant\xi\}$, $F^+:=\{x\in F\colon x\geqslant\xi\}$. Consider $(X\setminus F^-)\cap(-\infty,\xi)$ as the union of all members of the family $\mathcal{F}$ of disjoint open intervals with endpoints in F^-. Let α be a strictly decreasing map of F^- onto F^+ such that for every $I=(a,b)\in\mathcal{F}$ and $J:=(\alpha(b),\alpha(a))$ one has $(f(x)-x)(f(y)-y)<0$ for all $(x,y)\in I\times J$. Then for any such I and J, the continuous and strictly decreasing square iterative root of the function $f|_{I\cup J}$ depends on an arbitrary function.*

Proof. Without loss of generality we may assume that $f(x)<x$ for $x\in I$. Fix an arbitrary point $x_0\in I$ and an arbitrary continuous and strictly decreasing function $\varphi_0\colon[f(x_0),x_0]\to J$ such that $\varphi_0\circ f(x_0)=f\circ\varphi_0(x_0)$. With the aid of Theorem 5.3.1 we may extend φ_0 to a continuous and strictly decreasing solution $\gamma\colon I\to J$ of the equation

$$\gamma\circ f(x)=f\circ\gamma(x), \qquad x\in I; \tag{11.2.3}$$

moreover, $\lim_{x\to a}\gamma(x)=\alpha(a)$ and $\lim_{x\to b}\gamma(x)=\alpha(b)$. Put $Z:=I\cup J$. Then the function $\varphi: Z\to Z$ given by the formula

$$\varphi(x):=\begin{cases}\gamma(x) & \text{for } x\in I,\\ \gamma^{-1}\circ f(x) & \text{for } x\in J,\end{cases} \tag{11.2.4}$$

is strictly decreasing and continuous. Moreover, for $x\in X$, we have

$$\varphi^2(x)=\gamma^{-1}\circ f\circ\gamma(x)=f(x)$$

in view of (11.2.3) whereas, for $x\in J$,

$$\varphi^2(x)=\gamma\circ\gamma^{-1}\circ f(x)=f(x),$$

i.e. φ is a desired square iterative root of $f|_Z$. ■

Remark 11.2.2. In case φ is a (necessarily strictly) decreasing and continuous square iterative root of a surjective, continuous and strictly increasing self-mapping f of an open or closed real interval X, the assumptions of Theorem 11.2.5 are actually satisfied. In fact, let ξ be the unique fixed point of φ. Obviously, $\xi\in F$. Let $I=(a,b)\in\mathscr{F}$ and suppose that $f(x)<x$ for $x\in I$. Put $\alpha:=\varphi|_{F^-}$ and take a $y\in(\alpha(b),\alpha(a))=\varphi(I)$. Then $y=\varphi(x)$ for an $x\in I$ whence $f(y)-y=\varphi^2(\varphi(x))-\varphi(x)=\varphi(f(x))-\varphi(x)>0$.

Remark 11.2.3. The general strictly decreasing and continuous solution $\varphi: X\to X$ of equation (11.0.1) for $N=2$ with a surjective, continuous and strictly increasing map $f: X\to X$ may be obtained as follows. We take all points $\xi\in F$ and bijections $\alpha: F^-\to F^+$ fulfilling the conditions described in Theorem 11.2.5 on intervals $I\in\mathscr{F}$ and we construct a suitable solution φ_I on $I\cup J$. We extend α onto F by taking $\bar\alpha: F\to F$ given by $\bar\alpha|_{F^-}=\alpha$ and $\bar\alpha|_{F^+}=\alpha^{-1}$, so that $\bar\alpha=\bar\alpha^{-1}$. Then setting

$$\varphi(x)=\varphi_I(x) \text{ for } x\in I\cup J,\ I\in\mathscr{F}, \qquad \varphi(x)=\bar\alpha(x) \text{ for } x\in F,$$

we check easily that φ is a strictly decreasing square iterative root of f. The continuity of φ results now from Theorem 11.2.1.

Comments. The description of continuous iterative roots we have presented in this section is due to P. I. Haĭdukov [1] and M. Kuczma [5, (b)].

11.3 Monotonic C^r solutions

We have seen (Theorem 11.2.2) that strictly increasing and continuous self-mappings of a real interval always possess iterative roots (of all orders) having the same regularity properties. Unfortunately, this is no longer valid for C^r mappings, in general. The surprise disappears if we realize that the extending procedure applied in the proof of Theorem 11.2.2 preserves the regularity only in the interiors of the intervals considered and there is no

reason to expect higher regularity at the sticking points. Indeed, in many cases there exists no iterative root even of class C^1, although the given mapping is of class C^r and with strictly positive first derivative in the whole of its domain. We shall deal with C^1 iterative roots problem later on.

To establish any positive result, assume at first that φ is a C^r solution of the equation

$$\varphi^N=f \tag{11.3.1}$$

where f is a C^r function mapping a real interval X into itself. Differentiating both sides of equation (11.3.1) p times, $p\in\{1,\dots,r\}$, we come to an equality of the form

$$f^{(p)}(x)=P_p(\varphi'(x),\dots,\varphi'(\varphi^{N-1}(x));\dots;\varphi^{(p)}(x),\dots,\varphi^{(p)}(\varphi^{N-1}(x))),$$

$x\in X$, where P_p is a (uniquely determined) polynomial of $p\cdot N$ variables. This observation allows us to establish the following (Kuczma [31].

Theorem 11.3.1. *Assume $r\in\mathbb{N}$ and $N\in\mathbb{N}\setminus\{1\}$ to be fixed and let f be a C^r self-mapping of a real interval $X=(a,b|$, $-\infty\leqslant a<b\leqslant\infty$ such that $a<f(x)<x$ and $f'(x)>0$ for $x\in X$. Fix arbitrarily a point $x_0\in X$, put $x_N:=f(x_0)$ and choose arbitrary points $x_1>x_2>\dots>x_{N-1}$ from the interval (x_N,x_0). Then, for every strictly increasing C^r surjection $\varphi_i: [x_{i+1},x_i]\to[x_{i+2},x_{i+1}]$, $i\in\{0,\dots,N-2\}$ such that*

$$\varphi_i^{(p)}(x_{i+1})=\varphi_{i+1}^{(p)}(x_{i+1}) \quad \text{for } i\in\{0,\dots,N-3\} \text{ and } p\in\{1,\dots,r\} \tag{11.3.2}$$

(for $N=2$ this condition disappears) and

$$P_p(\varphi_0'(x_0),\dots,\varphi_{N-2}'(x_{N-2}),\varphi_{N-2}'(x_{N-1});\dots;\varphi_0^{(p)}(x_0),\dots,$$
$$\varphi_{N-2}^{(p)}(x_{N-2}),\varphi_{N-2}^{(p)}(x_{N-1}))=f^{(p)}(x_0) \quad \text{for } p\in\{1,\dots,r\}, \tag{11.3.3}$$

there exists a unique function $\varphi: X\to X$ satisfying (11.3.1) and such that $\varphi|_{[x_{i+1},x_i]}=\varphi_i$ for $i\in\{0,\dots,N-2\}$. This function is strictly increasing and of class C^r in X.

The proof is literally the same as that of Theorem 11.2.2. Assumptions (11.3.2) and (11.3.3) assure the C^r regularity of a solution constructed in that way. We omit the tedious although almost evident calculations.

Equally straightforward are the following two results.

Theorem 11.3.2. *Under the assumptions of Theorem 11.2.3, if, moreover, f is of class C^r, $1\leqslant r\leqslant\infty$, in Z with nowhere vanishing first derivative and if ψ is of class C^r in X, then so is the solution φ given by formula (11.2.2).*

Theorem 11.3.3. *Under the assumptions of Theorem 11.2.5, if, moreover, f is of class C^r, $1\leqslant r\leqslant\infty$, in $Z:=I\cup J$ with nowhere vanishing first derivative in Z and if γ is a C^r solution of equation (11.2.3) in I, then the function $\varphi: Z\to Z$ given by (11.2.4) yields a C^r and strictly decreasing square iterative root of $f|_Z$.*

Somewhat deeper results will be presented in the next two sections with regard to increasing and C^1 iterative roots. It turns out that their behaviour depends essentially on whether the multiplier of the given function (i.e. its derivative at the fixed point) vanishes or not. For that reason we shall emphatically distinguish between these two cases.

11.4 C^1 iterative roots with nonzero multiplier

11.4A. A function with no smooth convex square roots

By means of the example formerly announced (J. Ger [1], see also Kuczma [31]) we wish to exhibit two phenomena: (a) diffeomorphism with no smooth (square) iterative roots; (b) a strictly increasing convex mapping with no convex (square) iterative roots.

One function will serve for both purposes; that means that even a junction of these two regularity requirements helps nothing.

Example 11.4.1. Fix an $s \in (0, 1)$ and points $x_0 < x_1$ from $(0, 1)$ such that $x_1 < \sqrt{s} \cdot x_0/s$. Take any convex mapping $f \in C^1(\mathbb{R})$ and such that

$$f(x) = sx \quad \text{for } x \in (-\infty, x_0] \qquad \text{and} \qquad f(x) = \alpha x + \beta \quad \text{for } x \in [x_1, \infty),$$

where $\alpha := (1 - \sqrt{s} \cdot x_0)/(1 - x_1)$, $\beta := (\sqrt{s} \cdot x_0 - x_1)/(1 - x_1)$.

Suppose that a function $\varphi: \mathbb{R} \to \mathbb{R}$ satisfies the equation $\varphi^2 = f$ and φ is a C^1 function or φ is convex. From Theorem 11.4.2 (resp. 11.4.3) below we infer that $\varphi(x) = \sqrt{s} \cdot x$ for all $x \in [0, x_0]$. In particular,

$$\varphi(x_0) = \sqrt{s} \cdot x_0 = f(x_1) = \varphi^2(x_1). \tag{11.4.1}$$

Moreover, in the case where φ is convex, Theorem 11.3.1 implies that $\varphi|_{(0,1)}$ is a C^1 function as well. In both cases there exists a finite limit $\lim_{x \to 1-} \varphi'(x)$. This leads to a contradiction. To see this, observe that φ has to be strictly increasing since so is f and Theorem 11.1.1 holds. Thus the sequence $x_n := \varphi^{-n}(x_0)$, $n \in \mathbb{N}_0$, is well defined and the 'new' x_1 coincides with the 'old' one because of (11.4.1).

Further, $(x_n)_{n \in \mathbb{N}_0}$ tends to the nearest (and unique) fixed point of φ at the right of x_0, i.e. $x_n \to 1$ as $n \to \infty$ (note that $\varphi(1) = 1$ on account of Lemma 11.1.1 and there are no other fixed points of φ except that at zero). And yet, the sequence $(\varphi'(x_n))_{n \in \mathbb{N}}$ diverges. To see this, observe that

$$\varphi'(x)\varphi'(\varphi(x)) = f'(x) \geqslant s > 0, \qquad x \in (0, 1), \tag{11.4.2}$$

whence

$$\varphi'(x_{n+2}) = \frac{f'(x_{n+2})}{\varphi'(\varphi(x_{n+2}))} = \frac{f'(x_{n+2})}{\varphi'(x_{n+1})} = \frac{f'(x_{n+2})}{f'(x_{n+1})}\varphi'(x_n), \qquad n \in \mathbb{N}_0,$$

and since $x_{n+2} > x_{n+1} > x_1$, we have $f'(x_{n+2}) = f'(x_{n+1}) = \alpha$, $n \in \mathbb{N}_0$, and

$$\varphi'(x_{n+2}) = \varphi'(x_n), \qquad n \in \mathbb{N}_0.$$

On the other hand, since (11.4.1) holds, we have $\varphi(x_1) = x_0$, whence by (11.4.2) $\varphi'(x_1) = f'(x_1)/\varphi'(x_0) = \alpha/\sqrt{s} \neq \varphi'(x_0)$. Thus the sequence $(\varphi'(x_n))_{n \in \mathbb{N}_0}$ is oscillating and hence divergent. This proves that φ cannot be of class C^1 in any neighbourhood of 1. Obviously, φ is neither convex nor concave in consequence of the nonmonotonic behaviour of the sequence $(\varphi'(x_n))_{n \in \mathbb{N}_0}$.

11.4B. Necessary conditions

We shall place the unique fixed point of a given mapping at zero. Such a step has obviously a technical meaning only. We proceed with the following.

Theorem 11.4.1. *Let f be a self-mapping of a real interval $X = [0, a|$, $0 < a \leqslant \infty$, satisfying the conditions*

$$0 < f(x) < x \quad \textit{for } x \in X \setminus \{0\} \qquad \textit{and} \qquad f'(x) > 0 \quad \textit{for } x \in X.$$

If $\varphi: X \to X$ is a C^1 solution of the functional equation

$$\varphi^N = f, \tag{11.4.3}$$

then the infinite product

$$G(x, y) := \prod_{n=0}^{\infty} \frac{f'(f^n(x))}{f'(f^n(y))} \tag{11.4.4}$$

is convergent for all pairs $(x, \varphi(x))$ from the graph of φ whereas φ itself satisfies the differential equation

$$\varphi'(x) = s^{1/N} G(x, \varphi(x)), \qquad x \in X; \tag{11.4.5}$$

here $s := f'(0)$ is the (positive!) multiplier of f.

Proof. φ has to be strictly increasing. Indeed, otherwise, according to Theorem 11.1.1, φ would be strictly decreasing; this implies the existence of a (unique) fixed point $\xi \in X \setminus \{0\}$ of φ whence $f(\xi) = \xi$ which contradicts our assumption on f. Consequently, $\varphi'(x) \geqslant 0$ for $x \in X$. However, relation (11.4.3) implies that

$$\varphi'(x) \prod_{i=1}^{N-1} \varphi'(\varphi^i(x)) = f'(x), \qquad x \in X, \tag{11.4.6}$$

whence, in particular, it follows that, in fact, $\varphi'(x) > 0$ for all $x \in X$. Moreover, Lemma 11.2.1(a) implies the equality $\varphi(0) = 0$ whence $\varphi^i(0) = 0$ for $i \in \mathbb{N}$. Therefore, putting $x = 0$ in (11.4.6), we get the equality $\varphi'(0) = s^{1/N}$.

On the other hand, formula (11.4.6) implies the relation

$$\frac{f'(x)}{f'(\varphi(x))} = \frac{\varphi'(x)}{\varphi'(\varphi^N(x))} = \frac{\varphi'(x)}{\varphi'(f(x))}, \qquad x \in X,$$

and hence, for any $k\in\mathbb{N}$, we get

$$\prod_{n=0}^{k-1}\frac{f'(f^n(x))}{f'(f^n(\varphi(x)))}=\frac{\varphi'(x)}{\varphi'(f^k(x))},\qquad x\in X,$$

since f and φ are permutable. We complete the proof by letting k tend to infinity. ■

11.4C. Main existence theorem

Theorem 11.4.1 suggests that a C^1 iterative root (if it does exist) has to be unique. Actually, this has been proved, together with the existence, under some supplementary conditions on the given function (Bratman [1], Crum [1], Kuczma–A. Smajdor [3] and Zdun [9]). These results are collected in the following theorem whose proof yields a perfect occasion for a survey of natural applications of the results concerning the existence and uniqueness of solutions of the crucial iterative functional equations (i.e. that of Abel, Schröder and, more generally, linear equations) in various function classes.

Theorem 11.4.2. *Assume the hypotheses of Theorem 11.4.1 and put* $s:=f'(0)$.

(a) *If* $s=1$ *and either*

(1) f *is concave*

or

(2) $f'(x)=1-b(m+1)x^m+O(x^{m+\delta})$, $x\to 0$,

then equation (11.4.3) *has a unique* C^1 *solution in* X*; this solution is given by the formula*

$$\varphi(x)=\alpha^{-1}(\alpha(x)+1/N)\qquad \text{for } x\in X\setminus\{0\},\ \varphi(0)=0 \tag{11.4.7}$$

where α *is a principal solution of the Abel functional equation*:

$$\alpha(f(x))=\alpha(x)+1,\qquad x\in X\setminus\{0\}, \tag{A}$$

(b) *If* $s\in(0,1)$ *and either*

(3) f *is convex or* f *is concave or*

(4) $f'(x)=s+O(x^\delta)$, $x\to 0$,

then equation (11.4.3) *has a unique* C^1 *solution in* X*; this solution is given by the formula*

$$\varphi(x)=\sigma^{-1}(s^{1/N}\sigma(x)),\qquad x\in X, \tag{11.4.8}$$

where σ *is a nontrivial principal solution of the Schröder functional equation*:

$$\sigma(f(x))=s\sigma(x),\qquad x\in X. \tag{S}$$

Here m, b *and* δ *denote some positive constants.*

Proof. A straightforward verification shows that formulae (11.4.7) and (11.4.8) both give a solution of equation (11.4.3).

Part (a). Assume (1). Then Theorem 2.4.3 guarantees that the Abel equation (A) possesses the family of strictly decreasing and convex (principal; see Section 9.1) solutions given by the formula

$$\alpha(x)=c+\lim_{n\to\infty}\frac{f^n(x)-f^n(x_0)}{f^{n+1}(x_0)-f^n(x_0)},\qquad x\in X\setminus\{0\}, \tag{11.4.9}$$

where $c\in\mathbb{R}$ and $x_0\in X\setminus\{0\}$. The left and right derivatives of α do exist at each point of the interior of X. Both of them are increasing and satisfy the linear functional equation

$$\beta(f(x))=\frac{1}{f'(x)}\beta(x),\qquad x\in \operatorname{int} X. \tag{11.4.10}$$

Since $g:=1/f'$ is positive and increasing (f being concave has a decreasing derivative) and since, consequently $\inf_{x\in\operatorname{int}X} g(x)=\lim_{x\to 0} g(x)=s=1$, we may apply Theorem 2.3.1; therefore, our one-sided derivatives differ by a constant factor only. But convex mappings have differentiability points! Let $z\in\operatorname{int}X$ be a differentiability point of α. If we had $\alpha'(z)=0$, then we would have $\alpha'(x)=0$ for all $x\geqslant z$, $x\in\operatorname{int}X$ (recall that α is convex and decreasing) contradicting the strict monotonicity of α. Thus the proportionality factor has to be 1, i.e. α is differentiable on the whole of int X and $\alpha'(x)<0$ for all $x\in\operatorname{int}X$. Differentiable convex mapping is necessarily of class C^1. Hence α and, consequently, φ is of class C^1 in $X\setminus\{0\}$. It remains to prove that

$$\lim_{x\to 0}\varphi'(x)=1. \tag{11.4.11}$$

For observe that (11.4.7) implies

$$f(x)<\varphi(x)<x\qquad \text{for } x\in\operatorname{int}X \tag{11.4.12}$$

whence

$$1\leqslant\frac{\alpha'(\varphi(x))}{\alpha'(x)}\leqslant\frac{\alpha'(f(x))}{\alpha'(x)}=\frac{1}{f'(x)}$$

for any $x\in\operatorname{int}X$. Referring to (11.4.7) again we get $\varphi'(x)=\alpha'(x)/\alpha'(\varphi(x))$, $x\in\operatorname{int}X$, whence

$$f'(x)<\varphi'(x)\leqslant 1,\qquad x\in\operatorname{int}X,$$

from which we get (11.4.11) by letting x tend to zero.

To prove formula (11.4.7) and hence the uniqueness of φ observe that α' is a monotonic solution of equation (11.4.10) for which the function $g:=1/f'$

satisfies the condition

$$\lim_{n\to\infty} g(f^n(x)) = \frac{1}{f'(0)} = 1, \qquad x \in X.$$

Thus, we may apply Theorem 2.3.2 getting

$$0 > \alpha'(x) = c\prod_{n=0}^{\infty} \frac{g(f^n(x_0))}{g(f^n(x))} = c\prod_{n=0}^{\infty} \frac{f'(f^n(x))}{f'(f^n(x_0))}, \qquad x \in \operatorname{int} X,$$

whence (see (11.4.4))

$$G(x, y) := \prod_{n=0}^{\infty} \frac{f'(f^n(x))}{f'(f^n(y))} = \frac{\alpha'(x)}{\alpha'(y)} \qquad \text{for all } x, y \in \operatorname{int} X. \tag{11.4.13}$$

Now Theorem 11.4.1 gives ($s = 1$)

$$\varphi'(x) = \frac{\alpha'(x)}{\alpha'(\varphi(x))}, \qquad x \in \operatorname{int} X, \tag{11.4.14}$$

for any C^1 solution of equation (11.4.3). This says that taking such a solution, we have $(\alpha \circ \varphi)' = \alpha'$ and hence $\alpha \circ \varphi = \alpha + c_0$ for some $c_0 \in \mathbb{R}$ whence

$$\alpha(f(x)) = \alpha(\varphi^N(x)) = \alpha(x) + Nc_0, \qquad x \in \operatorname{int} X,$$

and since α satisfies Abel's functional equation (A) we infer that $c_0 = 1/N$ and, finally, $\varphi(x) = \alpha^{-1}(\alpha(x) + 1/N)$ for all $x \in X\setminus\{0\}$. That $\varphi(0) = 0$ results from Lemma 11.2.1(a).

Assume (2). Then Theorem 3.5.6 guarantees directly that α, given, again, by formula (11.4.9) is strictly decreasing and of class C^1 in $X\setminus\{0\}$. Moreover, from relation (3.5.26) ($r = -b^{-1}$) we learn that

$$\alpha'(x) = -\lim_{n\to\infty} b^{-1}(nbm)^{1+1/m}(f^n)'(x), \qquad x \in X\setminus\{0\},$$

whence

$$\frac{\alpha'(x)}{\alpha'(y)} = \lim_{n\to\infty} \frac{(f^n)'(x)}{(f^n)'(y)} = \lim_{n\to\infty} \prod_{i=0}^{n-1} \frac{f'(f^i(x))}{f'(f^i(y))} = G(x, y)$$

for all $x, y \in X\setminus\{0\}$, i.e. (11.4.13) remains valid; this, as previously, implies the uniqueness of φ. To show that φ is of class C^1 at the origin, recall that (11.4.14) holds true whence on account of (3.5.23)

$$\varphi'(x) = \frac{\alpha'(x)}{\alpha'(\varphi(x))} = \frac{x^{m+1}\alpha'(x)}{\varphi(x)^{m+1}\alpha'(\varphi(x))}\left(\frac{\varphi(x)}{x}\right)^{m+1}$$

$$= \frac{-b^{-1} + O(x^\tau)}{-b^{-1} + O(\varphi(x)^\tau)}\left(\frac{\varphi(x)}{x}\right)^{m+1}, \qquad x \to 0, \tag{11.4.14'}$$

where $\tau := \min(m, \delta)$. On the other hand, estimates (11.4.12) are satisfied leading to the relation

$$\frac{f(x)}{x} < \frac{\varphi(x)}{x} < 1, \qquad x \in X\setminus\{0\},$$

and therefore, since $f'(0) = 1$, we have

$$\varphi'(0) = \lim_{x\to 0} \frac{\varphi(x)}{x} = 1.$$

Now, letting $x \to 0$ in (11.4.14'), we get (11.4.11) which yields the the desired result.

Part (b). Assume (3). Then Theorem 2.4.4 guarantees that the Schröder equation (S) possesses the family of monotonic and convex (resp. concave) principal solutions (see Section 9.1) given by the formula

$$\sigma(x) = c\lim_{n\to\infty} \frac{f^n(x)}{f^n(x_0)}, \qquad x \in X, \tag{11.4.15}$$

where $c \in \mathbb{R}^+$ and $x_0 \in X\setminus\{0\}$. Except the trivial case $\sigma = 0$ (corresponding to the choice $c = 0$) these solutions are actually strictly monotonic. Indeed, if σ were (nonzero) constant on an interval $I = [u, v] \subset X$, it would be constant on each $f^n(I)$, $n \in \mathbb{N}$, violating the convexity (concavity) of σ because by (S) the constant mappings $\sigma|_I$ and $\sigma|_{f^n(I)}$ are different.

To prove the C^1 regularity of the function φ given by (11.4.8) on the whole of X (no matter which nontrivial solution (11.4.15) has been taken) it suffices to reproduce the reasoning applied for the proof in case (1); this time

$$\lim_{x\to 0} \varphi'(x) = s^{1/N},$$

whereas any nontrivial σ given by (11.4.15) is of class C^1 in $X\setminus\{0\}$ with $\sigma'(x) \neq 0$, $x \in X\setminus\{0\}$, and satisfies the functional equation

$$\sigma'(f(x)) = \frac{s}{f'(x)}\sigma'(x), \qquad x \in X\setminus\{0\}. \tag{11.4.16}$$

Making use of Theorem 2.3.2 once more we deduce that

$$\sigma'(x) = \tilde{c}\prod_{n=0}^{\infty} \frac{f'(f^n(x))}{f'(f^n(x_0))}, \qquad x \in X\setminus\{0\},$$

with some $\tilde{c} \neq 0$, whence (see (11.4.4))

$$G(x, y) = \frac{\sigma'(x)}{\sigma'(y)} \qquad \text{for all } x, y \in X\setminus\{0\}. \tag{11.4.17}$$

Now, Theorem 11.4.1 gives

$$\varphi'(x) = s^{1/N}\frac{\sigma'(x)}{\sigma'(\varphi(x))}, \qquad x \in X\setminus\{0\},$$

for any C^1 solution of equation (11.4.1). This says that taking such a solution we have $(\sigma \circ \varphi)' = (s^{1/N}\sigma)'$ and hence

$$\sigma(\varphi(x)) = s^{1/N}\sigma(x) + c_0$$

for some $c_0 \in \mathbb{R}$ and all $x \in X\setminus\{0\}$. Letting here $x \to 0$ we get $c_0 = 0$ in view of continuity of σ at 0, since obviously $\sigma(0) = \varphi(0) = 0$. Therefore formula (11.4.8) holds true.

Assume (4). Then Theorem 3.5.1 guarantees directly that the Schröder equation (S) has a C^1 solution σ in X given by the formula

$$\sigma(x)=\lim_{n\to\infty}\frac{f^n(x)}{s^n},\qquad x\in X.$$

Since σ' is a continuous solution of equation (11.4.16) we may apply Theorem 3.1.4 (which assures that 'case (A)' occurs) and, subsequently, Theorem 3.1.2, getting

$$\sigma'(x)=\sigma'(0)\lim_{n\to\infty}\prod_{i=0}^{n-1}\frac{f'(f^i(x))}{s},\qquad x\in X.$$

This leads again to formula (11.4.17) and the rest follows along the same lines as in the preceding case. ∎

It is noteworthy to observe that a powerful corollary follows.

Corollary 11.4.1. *In the circumstances described in Theorem 11.4.2 the function f is embeddable into an iteration semigroup (see Section 1.7) whose members are of class C^1.*

This result is just at hand: it suffices to replace the number $1/N$ in formulae (11.4.7) and (11.4.8) by a 'continuous' parameter $t\in\mathbb{R}^+$. This should come as no surprise. Our regularity assumptions imposed on f turned out to be strong enough to produce elegant and smooth iteration semigroups (see also Kuczma [26, Ch. IX]).

Remark 11.4.1. Assumptions (2) and (4) are of a local character. This is not the case for (1) and (3) and this might seem somewhat restrictive. As a matter of fact, the problem is apparent only. Actually, there is no need to assume the concavity (convexity) of f on the whole of its domain; it suffices to assume it merely in a right vicinity of zero. Then we obtain local C^1 iterative roots which, by Theorem 11.3.1, can be uniquely extended onto the whole interval considered.

11.4D. Convex and concave iterative roots

In Subsection 11.4A (Example 11.4.1) we were faced by a situation where a convex and smooth map admits no convex square iterative roots. Nevertheless, if an iterative root does exist (which, for instance, actually takes place if we assume the concavity of both f and f' (Theorem 11.4.4 below, Zdun [13]), then it has a representation (11.4.7) or (11.4.8) depending on whether $s=1$ or $s\in(0,1)$. More precisely, we have the following result of M. Kuczma–A. Smajdor [2].

Theorem 11.4.3. *Let f be a convex or concave self-mapping of an open interval*

$(0,a)\subset\mathbb{R}$. Suppose that $(1/x)f(x)\to s\in(0,1]$ as $x\to 0$, $f(x)<x$ for $x\in(0,a)$ and $f'>0$ in the domain of its existence. If a convex or concave Nth iterative root φ of f does exist, then

$$s=1\quad\textit{implies}\quad \varphi(x)=\alpha^{-1}(\alpha(x)+1/N),\qquad x\in(0,a),$$

where α is a principal solution of Abel's equation (A), whereas

$$s\in(0,1)\quad\textit{implies}\quad \varphi(x)=\sigma^{-1}(\sigma(x)+1/N),\qquad x\in(0,a),$$

where σ stands for a nontrivial principal solution of Schröder's equation (S).

The proof is similar to suitable parts of the proof of Theorem 11.4.2 and so we will omit it passing to Zdun's result announced previous to that.

Theorem 11.4.4. *Assume f to satisfy the assumptions of Theorem 11.4.1 and suppose additionally that both f and f' are concave. Then the unique C^1 iterative root (of a given order) of f is concave and this is the only concave iterative root (of that order) of f.*

Proof. Let $\varphi\colon X\to X$ be the unique C^1 solution of equation (11.4.3) (see Theorem 11.4.2). In particular, we have $\varphi'(x)>0$ and $f(x)\leqslant\varphi(x)\leqslant x$ for all $x\in X$. As f and φ are permutable we get

$$\frac{\varphi'(f(x))}{\varphi'(x)}=\frac{f'(\varphi(x))}{f'(x)}=:\psi(x),\qquad x\in X,$$

and, since f' decreases, ψ is greater than or equal to 1 whence

$$\varphi'(f(x))\geqslant\varphi'(x)\qquad\text{for all } x\in X.$$

Inductively, $\varphi'(f^n(x))\geqslant\varphi'(x)$, $x\in X$, $n\in\mathbb{N}$; letting n tend to infinity one obtains $\varphi'(x)\leqslant\varphi'(0)\in[s,1]$, i.e.

$$\varphi'(x)\leqslant 1\qquad\text{for all } x\in X. \tag{11.4.18}$$

The function $\lambda:=\log f'$ is evidently concave and decreasing (since, by assumption, f' is). Therefore, for $y\in X$ arbitrarily fixed and for any $x\in(\varphi(y),y)$ we have $\varphi(x)<\varphi(y)<x<y$ and hence

$$\frac{\lambda(y)-\lambda(x)}{y-x}\leqslant\frac{\lambda(x)-\lambda(\varphi(y))}{x-\varphi(y)}\leqslant\frac{\lambda(\varphi(y))-\lambda(\varphi(x))}{\varphi(y)-\varphi(x)};$$

consequently, for some $\xi\in(x,y)$, we get

$$\varphi'(\xi)(\lambda(y)-\lambda(x))=\frac{\varphi(y)-\varphi(x)}{y-x}(\lambda(y)-\lambda(x)\leqslant\lambda(\varphi(y))-\lambda(\varphi(x)).$$

In view of the fact that λ is decreasing and (11.4.18) holds true we come to the inequality

$$\lambda(y)-\lambda(x)\leqslant\lambda(\varphi(y))-\lambda(\varphi(x))$$

which says that

$$\frac{f'(y)}{f'(x)} \leqslant \frac{f'(\varphi(y))}{f'(\varphi(x))},$$

i.e. $\psi(x) \leqslant \psi(y)$. Finally, bearing the continuity of ψ in mind, what we have proved is the implication

$$x \in [\varphi(y), y] \quad \text{implies} \quad \psi(x) \leqslant \psi(y) \tag{11.4.19}$$

for any $y \in X$. Now, take any $u, v \in X$, $u < v$. Then, for some $n \in \mathbb{N}_0$, one has $u \in [\varphi^{n+1}(v), \varphi^n(v))$ whence, by (11.4.19),

$$\psi(u) \leqslant \psi(\varphi^n(v)) \leqslant \psi(\varphi^{n-1}(v)) \leqslant \cdots \leqslant \psi(v).$$

Thus ψ is increasing and, consequently, for any $x, y \in X$, $x < y$, we get

$$\frac{\varphi'(f(x))}{\varphi'(f(y))} \leqslant \frac{\varphi'(x)}{\varphi'(y)}$$

and, inductively,

$$\frac{\varphi'(f^n(x))}{\varphi'(f^n(y))} \leqslant \frac{\varphi'(x)}{\varphi'(y)} \qquad \text{for all } n \in \mathbb{N},$$

whence, as $n \to \infty$, we obtain $\varphi'(y) \leqslant \varphi'(x)$, i.e., φ' is decreasing and thus φ is concave. The uniqueness statement results from Theorem 11.4.3. ■

11.5 C^1 iterative roots with zero multiplier

As we have pointed out several times (see Chapters 0, 1, 8) the Böttcher functional equation

$$\beta(f(x)) = \beta(x)^p \tag{B}$$

occurs naturally and usually serves well when linearization via the Schröder (resp. the Abel) equation is not possible; the idea is to replace the linear mapping by a power function. The content of the present section may also be regarded as a step-by-step verification of the appropriateness of such an approach. All the results are due to M. Kuczma [33], [40].

11.5A. Abundance of solutions

Contrary to the case of mappings with nonvanishing multiplier, the C^1 iterative roots for functions with multiplier zero depend on an arbitrary function! (See Theorem 11.5.1 below.) So, the problem arises to choose a function class assuring the uniqueness. That one which fits well while we are considering Böttcher's equation is that ensuring a suitable asymptotic behaviour of the roots at the fixed point of the given map. This is described by Theorem 11.5.3 in detail.

We begin with a technical lemma whose standard proof will be omitted.

Lemma 11.5.1. *Let f be a C^1 self-mapping of an interval $X = [0, a|$, $0 < a \leqslant \infty$, such that $0 < f(x) < x$ and $f'(x) > 0$ for all $x \in (0, a|$ and $f'(0) = 0$ (multiplier zero). Then, for any constant $p > 1$, the function*

$$F(x) := \begin{cases} x^{-p} f(x) & \text{for } x \in X \setminus \{0\}, \\ b & \text{for } x = 0, \end{cases}$$

is of class C^1 if and only if

$$f'(x) = bpx^{p-1} + c(p+1)x^p + o(x^p), \qquad x \to 0, \tag{11.5.1}$$

for some constants $b > 0$ and $c \in \mathbb{R}$.

Suppose now that f satisfies the assumptions of Lemma 11.5.1 and $\varphi: X \to X$ is a C^1 solution of the equation

$$\varphi^N = f. \tag{11.5.2}$$

Then $\varphi'(x) > 0$ for $x \in X \setminus \{0\}$ and $\varphi'(0) = 0$; moreover, from the proof of Theorem 11.4.1 we learn that

$$\varphi'(f^k(x)) = \varphi'(x) \prod_{n=0}^{k-1} \frac{f'(f^n(\varphi(x)))}{f'(f^n(x))}, \qquad x \in X, \tag{11.5.3}$$

for any $k \in \mathbb{N}$. Since the sequence $(f^k)_{k \in \mathbb{N}}$ of iterates of f tends to zero almost uniformly on X (see Theorem 1.2.2), the convergence

$$\lim_{k \to \infty} \prod_{n=0}^{k-1} \frac{f'(f^n(y))}{f'(f^n(x))} = 0 \tag{11.5.4}$$

is almost uniform in (x, y) from the graph of the map $\varphi|_{(0,a|}$.

A (partially) converse result holds true.

Theorem 11.5.1. *Assume the hypotheses of Lemma 11.5.1 and fix an $x_0 \in (0, a|$. If the convergence (11.5.4) is almost uniform in (x, y) from the set*

$$\{(x, y) \in [f(x_0), x_0] \times \mathbb{R} : f(x) < y < x\}, \tag{11.5.5}$$

then the C^1 iterative roots of f on $[0, a|$ form a family depending on an arbitrary function.

Proof. In view of Theorem 3.1, the assertion is obviously true on the interval $(0, a|$. The point is to extend any C^1 iterative root φ of $f|_{(0,a|}$ to a C^1 solution of equation (11.5.2) on the whole of $[0, a)$. To this aim observe that for any $x \in [f(x_0), x_0]$ the pair $(x, \varphi(x))$ belongs to set (11.5.5) (see Lemmas 11.2.2 and 11.2.1) which jointly with relation (11.5.3) and the assumed a.u. convergence (11.5.4) implies

$$\lim_{k \to \infty} \varphi'(f^k(x)) = 0$$

uniformly on the interval $[f(x_0), x_0]$. Consider any sequence $(y_n)_{n \in \mathbb{N}}$

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MAREK KUCZMA
Institute of Mathematics, Silesian University, Katowice, Poland

BOGDAN CHOCZEWSKI
Institute of Mathematics, Academy of Mining and Metallurgy, Kraków, Poland

ROMAN GER
Institute of Mathematics, Silesian University, Katowice, Poland

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