ON ANALYTIC ITERATION

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1. Definitions.

Let Ω be the set of all analytic functions $F(z)$ which admit an expansion of the type:

$$F(z) = z + f_2 z^2 + f_3 z^3 + \dots,$$

convergent for:

$$|z| < \rho \quad , \quad \rho > 0 .$$

Let S be a set of complex numbers such that $a \in S$ and $b \in S$ implies $(a - b) \in S$, with $1 \in S$.

The function $F(z)$ will be said *to possess iterates* in S if there exists a function $F(s, z)$, called the s-iterates of $F(z)$, defined for $s \in S$, satisfying the following four conditions:

(1) $F(1, z) = F(z)$,

(2) $F(s, z) \in \Omega$, $(s \in S)$,

(3) $F[s, F(s', z)] = F[(s + s'), z]$, $(s, s' \in S)$,

(4) $F(s, z) = \sum_{k=1}^{k=\infty} f_k(s) z^k$ (for $s \in S$, $|z| < \rho(s)$, $\rho(s) > 0$),

where $f_k(s)$ are polynomials in s.

If the set S is the set of all integers, $F(s, z)$ is said to be *the integer iterate* of $F(z)$.

If the set S is the set of all real numbers, $F(s, z)$ is said to be *the complete real iterate* of $F(z)$.

If the set S is the set of all complex numbers, $F(s, z)$ is said to be *the complete complex iterate* of $F(z)$.

If $F(s, z)$ is a complete complex iterate of $F(z)$ and is analytic in s, it is said to be *the analytic iterate* of $F(z)$.

2. The Main Theorem.

The purpose of this paper is to prove that:

If the function $F(z) \in \Omega$ admits a complete real iterate then a function $F(s, z)$ exists which is the complete complex iterate of $F(z)$. This function is analytic in s and is therefore the analytic iterate of $F(z)$. If $F(z)$ does not have an analytic iterate, then it can only have iterates $F(s, z)$ in a real set S of one-dimensional measure zero and in a complex set S of two-dimensional measure zero.

This theorem partially fills the gap in our knowledge about the analiticity (in s) of the s-iterates of analytic functions. Indeed, it is well known [4], [6] that functions of the type $F(z) = f_1 z + f_2 z^2 + \ldots$ with $|f_1| \neq 1$ always have complete complex and analytic iterates. The case $|f_1| = 1$, but $f_1 \neq 1$ is still largely open [4]. The case $f_1 = 1$ is the one covered by our Theorem.

The theorem shows that the functions with $f_1 = 1$ fall into two complementary classes: those having a complete complex and analytic iterate and those who have iterates only in a set S of one- or two-dimensional measure zero. The two classes are not void. The function

$$F(s, z) = \frac{z}{1 - sz}$$

is a classical example of an analytic iterate. The function $e^z - 1$ was shown by I. N. Baker [1] to have no real non-integer iterates. M. Levine [6] showed, using some results of the present paper, that this function and the functions $z + z^2$ and $\frac{z}{(1-z)^2}$ have no analytic iterate.

To prove our theorem we need some classical results from the theory of integer iterates of functions in Ω.

3. The integer iterates.

If S is the set of all integers, the condition (4) in our definition of $F(s, z)$ is redundant. More precisely we have the following well known result which we quote in a form convenient for us and prove for completeness' sake:

Let $F(z) \in \Omega$ have the expansion:

$$F(z) = z + f_2 z^2 + f_3 z^3 + \dots \qquad (\text{for } |z| < \rho \, , \, \rho > 0) \, .$$

Then the s-iterate $F(s, z)$ of $F(z)$, for integer s, which satisfies conditions (1), (2) and (3) is uniquely defined and can be expanded in the power series:

$$F(s, z) = \sum_{k=1}^{k=\infty} f_k(s) z^k \qquad (\text{for } |z| < \rho(s) \, , \, \rho(s) > 0) , \tag{5}$$

where the functions $f_k(s)$ are polynomials in s (so that condition (4) is automatically satisfied) of degree $n \leqq k-1$. Furthermore:

$$f_k(0) = \delta_{k,i} \; ; \; f_k(1) = f_k \; ; \; f_1(s) = 1 \tag{6}$$

and the degree of $f_k(s)$ is $n \leqq k - 1$.

Proof [3]: Let $m > 0$ be an integer. Expand $[F(z)]^m$ in a power series and put:

$$[F(z)]^m = \sum_{k=1}^{k=\infty} f_{m,k} z^k \qquad (\text{with } f_{m,k} = 0 \text{ for } k < m) .$$

Consider now the matrix:

$$\mathbf{F} = \| f_{m,k} \| , \qquad (m = 1, 2, \dots ; \; k = 1, 2, \dots) .$$

It is readily shown by induction, using condition (3) that, for positive integer s and m, if we write:

$$[f(s, z)]^m = \sum_{k=1}^{k=\infty} f_{m,k}(s) z^k$$

and:

$$\mathbf{F}(s) = \| f_{m,k}(s) \| , \qquad (m = 1, 2, \dots ; \; k = 1, 2, \dots)$$

then:

$$\mathbf{F}(s) = (\mathbf{F})^s \qquad (\text{equation between matrices}). \tag{7}$$

Noting that the matrix $\mathbf{F}$ is triangular, with all its diagonal elements $= 1$, and denoting by $\mathbf{J}$ the unit matrix:

$$\mathbf{J} = \| \delta_{m,k} \| , \qquad (m = 1, 2, \dots ; \; k = 1, 2, \dots) ,$$

we have, using the fact that the unit matrix commutes with all matrices:

$$\mathbf{F}(s) = [(\mathbf{F}-\mathbf{J}) + \mathbf{J}]^s = \sum_{\sigma=0}^{\sigma=s} \binom{s}{\sigma} (\mathbf{F}-\mathbf{J})^\sigma \qquad (\text{where } (\mathbf{F}-\mathbf{J})^0 = \mathbf{J}).$$

Let $(\mathbf{F}-\mathbf{J})^\sigma_{m,k}$ be the element of the m-th row and the k-th column of the matrix $(\mathbf{F}-\mathbf{J})^\sigma$, then:

$$(\mathbf{F}-\mathbf{J})^\sigma_{m,k} = 0 \quad \text{for} \quad \sigma > k-m$$

because the main diagonal of the matrix $(\mathbf{F}-\mathbf{J})$ is zero.

It follows that $f_k(s)$, which is the element $(1, k)$ of the matrix $\mathbf{F}(s)$, is given by:

$$f_k(s) = \sum_{\sigma=0}^{\sigma=s} \binom{s}{\sigma} (\mathbf{F}-\mathbf{J})^\sigma_{1,k} = \sum_{\sigma=0}^{\sigma=k-1} (\mathbf{F}-\mathbf{J})^\sigma_{1,k} \binom{s}{\sigma}. \tag{8}$$

Thus $f_k(s)$ is a polynomial in s. The highest degree of s occurs in the term with the highest σ. This degree is thus $(k-1)$ or less (if $(\mathbf{F}-\mathbf{J})^{k-1}_{1,k} = 0$).

Noting that $\binom{s}{\sigma} = 0$ if $\sigma > s$ we now easily verify conditions (6).

4. Non integer s.

We now have to examine our definition of $F(s, z)$ for non integer s. Conditions (1) and (3) are unavoidable in any extension of the definition of iteration. Condition (2) is arbitrary but seems to be a natural requirement without which the problem would be too indefinite.

Condition (4), for non integer s, does not result from conditions (1), (2) and (3). Indeed, let $f_k(s)$ be defined by (8) and suppose conditions (1), (2) and (3) to be satisfied. Let $H(s)$ be a Hammel function defined for $s \in S$. That is let:

$$H(s) \not\equiv s \, , \; H(1) = 1$$

and

$$H(s+s') = H(s) + H(s'), \qquad (s, s' \in S).$$

Then the function:

$$F^*(s\,,\,z) = \sum_{k=1}^{k=\infty} f_k\,[H(s)]\,z^k\,,$$

which, for integer s coincides with $F(s\,,\,z)$, satisfies conditions (1), (2) and (3) for all $s \in S$, but not condition (4).

Condition (4) is therefore necessary to ensure the unicity of the function $F(s\,,\,z)$ for those values of s for which it is defined. It is also sufficient to ensure this unicity because a polynomial $f_k(s)$ is uniquely determined if its values are given for all integer s.

Note that condition (4) could be replaced, in the case where the set S is dense, by the requirement that the functions $f_k(s)$ be continuous functions of s.

5. The sequence $\{l_k\}$.

Consider the sequence of numbers $\{l_k\}$ defined by:

$$l_k = f'_{k+1}\,(0)\,.$$

(Note that we could as well have written l_{k+1} for l_k. Our choice of notation is made to conform with other usage). Here $f'_k\,(s)$ is the derivative of the polynomial $f_k(s)$ defined by equation (8), so that:

$$(9) \qquad l_0 = 0\,; \quad l_k = \sum_{\sigma=1}^{\sigma=k} \frac{(-1)^{\sigma-1}}{\sigma}\,(\mathbf{F}-\mathbf{J})^{\sigma}_{1\,,\,k+1}\,, \qquad (k = 1,\,2,\,\ldots)\,.$$

Equations (9) show that the function $F(z)$ determines uniquely the sequence $\{l_k\}$.

Conversely, the sequence $\{l_k\}$ determines uniquely the sequence $\{1\,, f_2\,, f_3\,, \ldots\}$ of the coefficients of the expansion of the function $F(z) \in \Omega$ which generated the sequence $\{l_k\}$.

Indeed the f_i with the highest subscript which appears in equations (9) is f_{k+1} and this appears only in the term for $s = 1$. In that term it appears with the coefficient 1. Therefore, writing equations (9) successively for $k = 1$, $k = 2, \ldots$, we can solve them successively and determine the numbers l_k. Clearly equations (9) cannot yield the coefficient of z in $F(z)$ but this coefficient is known to be 1 because $F(z) \in \Omega$.

6. The function $L(z)$.

Consider the formal expansion:

$$L(z) = \sum_{k=1}^{k=\infty} f'_k(0) z^k = \sum_{k=2}^{k=\infty} l_{k-1} z^k . \tag{10}$$

There are two cases according to whether the radius of convergence ρ of the series on the right is positive or is zero.

We note that if $F(z)$ has an analytic iterate $F(s, z)$ then:

$$L(z) = \left. \frac{\partial F(s, z)}{\partial s} \right|_{s=0} .$$

This results from definitions (4) and (10). Furthermore the function $L(z)$ also satisfies the double functional-differential equation [4]:

$$L[F(s, z)] = \frac{\partial F(s, z)}{ds} = L(z) \cdot \frac{\partial F(s, z)}{\partial z} . \tag{11}$$

Indeed, differentiating equation (3) over s' we find:

$$\frac{\partial F(s', z)}{\partial s'} = F_z[s, F(s', z)] = F_s[(s + s'), z] .$$

Putting $s' = 0$ and noting that, by (6), $F(0, z) = z$, we get the second half of equation (11). Differentiating equation (3) over s we find:

$$F_s[s, F(s', z)] = F_s[(s + s'), z]$$

Putting $s = 0$ and changing s' into s we get the first half of equation (11).

All this holds however only if the function $F(z)$ has an analytic iterate. This is not always the case.

We shall prove two theorems, corresponding to the two possibilities, which together will be seen to be equivalent to our Main Theorem.

7. Two theorems.

Theorem I. *If the radius of convergence of the series (10) is $\rho > 0$ then the series defines a function $L(z)$ and permits to construct a uniquely defined function $F(s, z)$ satisfying conditions (1) to (4) for $|z| < \rho(s)$ with $\rho(s) > 0$ for all finite complex s. This function $F(s, z)$ is then analytic* in s and in z for all finite complex s and for $|z|<\rho(s)$. It is the complete complex iterate of $F(z)$ and is also the analytic iterate of $F(z)$.

Theorem II. If the radius of convergence of the series in (10) is $\rho=0$, then the radius of convergence $\rho(s)$ of the series $\sum_{k=1}^{k=\infty} f_k(s)s^k$ is $=0$ for almost all complex s and for almost all real s. The function $F(z)$ then does not admit a complete complex or a complete real iterate.

As there are only two possibilities, the function $F(s, z)$ qua function of s must either be defined for every finite complex s and be analytic in s, or it must exist only for real values of s belonging to a set of one-dimensional measure zero or for complex values of s belonging to a set of two-dimensional measure zero, which is our Main Theorem.

8. Proof of Theorem I.

We assume that the radius of convergence of the series (10) is $\rho>0$ and propose to prove that then there exists a unique function $F(s, z)$ satisfying conditions (1) to (4) and that this function is analytic in s.

Consider the differential equation

$$\frac{d\zeta}{dz}=\frac{L(\zeta)}{L(z)}=\frac{l_1\zeta^2+l_2\zeta^3+\ldots}{l_1z^2+l_2z^3+\ldots}. \tag{12}$$

This equation has a meaning for $|z|, |\zeta|<\rho$ but it has a singularity for $z=\zeta=0$ so that Cauchy's existence theorem is not directly applicable to its solution in the neighborhood of $z=0$. Let the first l_k which is not $=0$ be l_p and put:

$$\zeta=z+z^{p+1}\eta.$$

Equation (12) becomes:

$$1+(p+1)z^p\eta+z^{p+1}\frac{d\eta}{dz}=\frac{L(z+z^{p+1}\eta)}{L(z)},$$

or:

$$\frac{d\eta}{dz}=\frac{L(z+z^{p+1}\eta)-[1+(p+1)z^p\eta]\,L(z)}{z^{p+1}L(z)}.$$

A short computation based on Taylor's theorem shows immediately that the right hand side is analytic for $\eta = \eta_0$ for any finite complex η_0 and for sufficiently small z. (Indeed for z such that $|z| < \rho$ and $|z + z^{p+1}\eta_0| < \rho$, where η_0 is any finite complex number). Cauchy's existence theorem (*) is now applicable and shows that the equation in η has a unique solution satisfying the initial conditions:

$$z = 0 \ ; \ \eta = \eta_0 .$$

We shall now choose $\eta_0 = l_p s$, where s is another arbitrary finite complex constant.

Equation (12) has therefore a unique solution of the form:

$$\zeta = \zeta(s, z) = z + f'_{p+1}(0)\, s z^{p+1} + z^{p+2}\xi(s, z), \tag{13}$$

where $\xi(s, z)$ is analytic in z for any s and for $|z| < \rho(s)$ for some $\rho(s) > 0$.

We now want to show that $\zeta(s, z) = F(s, z)$ and is analytic in s. The proof of this is quite straightforward and elementary but is somewhat lengthy as we have to show that each of the conditions (1) to (4) is satisfied.

Condition (2) is satisfied as, by (13), $\zeta(s, z) \in \Omega$.

To show that condition (3) is satisfied, consider $\zeta(s + s', z)$ and $\zeta(s', z)$. These functions, qua functions of z, satisfy the equations:

$$\frac{d\zeta(s + s', z)}{dz} = \frac{L[\zeta(s + s', z)]}{L(z)}$$

and

$$\frac{d\zeta(s', z)}{dz} = \frac{L[\zeta(s', z)]}{L(z)},$$

with constant s and s'. Whence by division:

$$\frac{d\zeta(s + s', z)}{d\zeta(s', z)} = \frac{L[\zeta(s + s', z)]}{L[\zeta(s', z)]}.$$

But this is an equation of type (12), its solutions are:

$$\zeta(s + s', z) = \zeta[s'', \zeta(s', z)],$$

* E. Goursat, Cours d'Analyse Mathématique. Tome II, p. 347, Paris 1911.

with constant s''. It remains to prove that $s'' = s$. This is seen by expanding both sides into powers of z by (13) and equating the coefficients of z^{p+1}.

To prove that condition (4) is satisfied, that is that the coefficients of the powers of z are polynomials in s, we proceed by induction. By (13) the proposition is true for the coefficients of z^k with $k \leqq p + 1$.

Put

$$\zeta(s, z) = \sum_{k=1}^{k=\infty} g_k(s) z^k$$

and carry this expansion into equation (3). Equating the coefficients of z^σ on both sides of (3) we find, assuming all previous coefficients to be polynomials:

$$g_\sigma(s + s') = g_\sigma(s) + s' P(s, s'),$$

where $P(s, s')$ is a polynomial in s and s'.

Therefore:

$$g'_\sigma(s) = P(s, 0)$$

and $g_\sigma(s)$ is also a polynomial in s.

The function $\zeta(s, z)$ has thus been shown to be the s-iterate of $\zeta(1, z)$. It remains to be shown that it is analytic in s and that condition (1) is satisfied so that $\zeta(1, z) = F_1(z)$.

For $|z| < \rho(s)$ the function $\zeta(s, z)$ is the sum of an absolutely convergent series of analytic functions of s (actually of the polynomials $z^k g_k(\varphi)$). The function $\zeta(s, z)$ is thus analytic in s and it remains to be shown that it is the analytic iterate of $\zeta(1, z)$.

We shall now show that

$$\left.\frac{\partial \zeta(s, z)}{\partial s}\right|_{s=0} = L(z).$$

Put

$$\left.\frac{\partial \zeta(s, z)}{\partial s}\right|_{s=0} = M(z).$$

By equations (11) we have:

$$\frac{\partial \zeta(s, z)}{\partial s} = M[\zeta(s, z)].$$

By equation (12) we have:

$$L(z) \cdot \frac{\partial \zeta(s, z)}{\partial s} = L[\zeta(s, z)].$$

Differentiating this over s we find:

$$L(z) \cdot \frac{\partial}{\partial z}\left[\frac{\partial \zeta(s, z)}{\partial s}\right] = L'[\zeta(s, z)] \cdot \frac{\partial \zeta(s, z)}{\partial s}.$$

Putting $s = 0$ and noting that $\zeta(0, z) = z$ (indeed z is a solution of equation (12) and is therefore the unique solution when $s = 0$), we get:

$$L(z) \cdot M'(z) = L'(z) \cdot M(z).$$

Therefore $M(z) = cL(z)$ where c is a constant. Using (13) we see that the coefficient of z^{p+1} in $M(z)$ is $f'_{p+1}(0) = l_p$, which is the coefficient of z^{p+1} in $L(z)$ so that $c = 1$ and:

$$\left.\frac{\partial \zeta(s, z)}{\partial s}\right|_{s=0} = L(z).$$

But we have seen that $L(z)$ can originate in this way from one function $F(z)$ only, so that:

$$\zeta(1, z) = F(z).$$

and

$$\zeta(s, z) = F(s, z)$$

is the complete complex and the analytic iterate of $F(z)$.

9. Two lemmas.

To prove Theorem II we need two lemmas.

Lemma 1 (real case): Let A and B be given positive numbers with $B \leqq A$. Let $p(x)$ be any polynomial of degree n in x with $|p(0)| \geqq A^k$ where $k \geqq n+1$. Then there exists an absolute constant c such that the one-dimensional measure m_1, of the set in real x for which $|x| < t$ and $|p(x)| < B^k$ is:

$$m_1 \leqq \frac{ctB}{A}.$$

Proof: We may suppose $n \geqq 1$, for if $n = 0$ then $m_1 = 0$.

Choose any $(n+1)$ numbers $x_1, x_2, \ldots, x_{n+1}$. Then by Lagrange's interpolation theorem:

$$p(x) = \sum_{i=1}^{i=n+1} \frac{s(x)}{s'(x_i)(x - x_i)} p(x_i),$$

where

$$s(x) = \prod_{j=1}^{j=n+1} (x - x_j)$$

and therefore

$$s'(x_i) = \prod_{j \neq i} (x_i - x_j).$$

We have:

$$(14) \qquad |p(0)| \leqq \sum_{i=1}^{i=n+1} \frac{|s(0)|}{|s'(x_i)| \cdot |x_i|} |p(x_i)|.$$

Now choose the $(n+1)$ numbers x_i so that:

$$(15) \qquad x_i = \text{real}; \quad \frac{\alpha}{A} < |x_i| < t; \quad x_{i+1} - x_i > \frac{\alpha}{nA}; \quad |p(x_i)| < B^k,$$

where α is a positive number to be chosen presently.

If we cannot find $(n+1)$ numbers satisfying (15), then all numbers x_i such that $|p(x_i)| < B^k$ are confined to, at most, n intervals of length $\frac{\alpha}{nA}$. Thus, in this case:

$$(16) \qquad m_1 \leqq n \frac{\alpha}{nA} = \frac{\alpha}{A}.$$

If, on the other hand, the $(n+1)$ numbers x_i can be found to satisfy (15), then:

$$A^k \leqq |p(0)|, \quad |s(0)| < t^{n+1}; \quad \frac{1}{|x_i|} < \frac{A}{\alpha} \quad \text{and} \quad |p(x_i)| < B^k.$$

Moreover we can estimate $|s'(x_i)|$ by noting that it takes its smallest value when the x_j are the closest to each other, that is when

$$x_{j+1} - x_j = \frac{\alpha}{nA}$$

and then for

$$i = \left[\frac{n+1}{2}\right].$$

Therefore:

$$\frac{1}{|s'(x_i)|} \leqq \frac{1}{\left\{\left[\frac{n+1}{2}\right]!\right\}^2 \left(\frac{\alpha}{nA}\right)^n} = \frac{n^n}{\left\{\left[\frac{n+1}{2}\right]!\right\}^2} \cdot \frac{A^n}{\alpha^n}.$$

Carrying all these estimates into (14) we get:

$$A^k < t^{n+1} \cdot \frac{A}{\alpha} \cdot B^k \cdot \frac{n^n}{\left[\left\{\frac{n+1}{2}\right]!\right\}^2} \cdot \frac{A^n}{\alpha^n}.$$

Let c_1 be the upper bound of $2\left[\frac{n^n}{\left\{\left[\frac{n+1}{2}\right]!\right\}^2}\right]^{1/(n+1)}$ then the above inequality yields:

$$\alpha < c_1 tA\left(\frac{B}{A}\right)^{k/(n+1)} = c_1 tB\left(\frac{B}{A}\right)^{(k-n-1)/(n+1)} \leqq c_1 tB,$$

because $B \leqq A$ and $k \geqq n+1$.

Choosing $\alpha = c_1 tB$ we arrive at a contradiction so that there are no $(n+1)$ numbers x_i satisfying (15) for this value of α. Therefore, from (16):

$$m_1 \leqq \frac{c_1 tB}{A},$$

which proves Lemma 1.

Lemma 2 (complex case): Let A and B be given positive numbers with $B \leqq A$. Let $p(x)$ be a polynomial of degree n in x with $|p(0)| \geqq A^k$ where $k \geqq n+1$. Then for all complex x there exists an absolute constant c' such that the two-dimensional measure m_2 of the set in complex x for which $|x| < t$ and $|p(x)| < B^k$ is:

$$m_2 \leqq \frac{c' t^2 B}{A}.$$

Proof: As in the proof of Lemma 1 we choose $(n+1)$ numbers x_i but now we demand that these numbers satisfy the conditions:

$$(17) \qquad \sqrt{\frac{\alpha t}{A}} < |x_i| < t; \quad |x_{i+1}| - |x_i| > \frac{\alpha}{nA}\,^{(*)}; \quad |p(x_i)| < B^k,$$

where α is a positive number to be determined presently. If we cannot find $(n+1)$ such numbers, then all the numbers x for which $|p(x)| < B^k$ are concentrated, at most, in n rings of width $\frac{\alpha}{nA}$ and outer radius $\leqq t$, so that, in this case:

$$(18) \qquad m_2 \leqq n \cdot 2\pi \frac{\alpha}{nA} t = 2\pi \frac{\alpha t}{A}.$$

Proceeding as in the proof of Lemma 1, we find that if $(n+1)$ numbers x_i satisfying (17) exist then:

$$A^k < t^{n+1} \cdot \sqrt{\frac{A}{\alpha t}} \cdot B^k \cdot \frac{n^n}{\left\{\left[\frac{n+1}{2}\right]!\right\}^2} \cdot \frac{A^n}{\alpha^n}.$$

Let c_2 be the upper bound of $\left[\frac{n^n \cdot 2^{n+1}}{\left\{\left[\frac{n+1}{2}\right]!\right\}^2}\right]^{1/(n+1/2)}$, then the above inequality yields:

$$\alpha < c_2 t A \left(\frac{B}{A}\right)^{k/(n+1/2)} = c_2 t B \left(\frac{B}{A}\right)^{(k-n-1/2)/(n+1/2)} \leqq c_2 t B,$$

as before.

Choosing $\alpha = c_2 t B$ we arrive at a contradiction so that there are no $(n+1)$ numbers x_i satisfying (17) for this value of α. Therefore, from (18):

$$m_2 \leqq \frac{2\pi c_2 t^2 B}{A},$$

which proves Lemma 2.

* We could replace this inequality by $|x_{i+1} - x_i| > \alpha/nA$, then by slightly longer computation we would obtain $m_2 < c'' t B/A$.

10. Proof of Theorem II.

We consider the sequence of polynomials

$$p_k(x) = \frac{1}{x} f_k(x),$$

Let n_k be the degree in x of $p_k(x)$. If $f_2 \neq 0$ then, by (6), $n_k \leqq k-2$. If $f_2 = 0$ the degree n_k is still smaller. In all cases $k > n+1$ and Lemmas 1 and 2 are applicable. As $f_k(0) = 0$ (for $k > 1$), we see that:

$$p_k(0) = f'_k(0) \qquad (\text{for } k \geqq 2).$$

Our assumption is that the series $\sum_{k=2}^{k=\infty} f'_k(0) z^k$ diverges for all $z \neq 0$, that is that, for any given $A > 0$ we have:

$$|p_k(0)| > A^k,$$

for infinitely many k.

Let B be any given positive number. Choose an increasing sequence of positive numbers A_q tending to infinity with q and such that $B \leqq A_1 < A_2 < \ldots$. It results from our assumption that, given any q, an integer k_q can be found such that:

$$| p_{k_1}(0) > A_q^{k_q}.$$

We have to prove that, given B, the one- (or two-) dimensional measure m_1 (or m_2) of the set in real (or complex) x for which:

$$\overline{\lim_{q\to\infty}} | p_{k_q}(x) |^{1/k_q} < B, \tag{19}$$

is zero.

It suffices to show this for $|x| < t$. Let $S^{(q)}_{t,B}$ (or $S'^{(q)}_{t,B}$) denote that set in real (or complex) x for which $|p_{k_q}(x)| < B^{k_q}$. Then, by our Lemmas:

$$m_1(S^{(q)}_{t,B}) \leqq \frac{cBt}{A_q} \quad ; \quad m_2(S'^{(q)}_{t,B}) \leqq \frac{c'Bt^2}{A_q}.$$

If x satisfies inequality (19) then $x \in S^{(q)}_{t,B}$ (or $x \in S'^{(q)}_{t,B}$) for all but a finite number of q, so that:

$$x \in \bigcup_{l=1}^{l=\infty} \bigcap_{q=l}^{q=\infty} S^{(q)}_{t,B} \qquad \left(\text{or } x \in \bigcup_{l=1}^{l=\infty} \bigcap_{q=l}^{q=\infty} S'^{(q)}_{t,B}\right). \tag{20}$$

But as $A_q \to \infty$ we have $m_1(S^{(q)}_{t,B}) \to 0$ (or $m_2(S'^{(q)}_{t,B}) \to 0$). Thus the measure of the sets in x satisfying (20) is zero.

11. General remarks.

Lemmas I and II are akin to H. Cartan's theorem [2]:

Let $h(z) = \prod_{j=1}^{n}(z-z_j)$ be a polynomial and $H>0$ any given positive number. Then $|f(z)| \geqq \left(\frac{H}{e}\right)^n$ everywhere except in a set covered by, at most, n circles of radius $r_1, r_2, \ldots, r_n$ such that $\sum_{i=1}^{n} r_i \leqq 2H$.

Our lemmas can be modified in many ways. In particular in Lemma II it is possible to obtain $m_2 \leqq \frac{c''tB}{A}$ instead of our $m_2 \leqq \frac{c't^2B}{A}$. We have prefered giving the weaker result, which is sufficient for our purposes, because the proof of Lemma II then becomes a repetition of that of Lemma I and is shorter. Similarly if we had taken $k = n+1$ instead of $k \geqq n+1$ the condition $A>B$ could be discarded. However, we needed the case $k>n+1$ because when $f_2 = 0$, the polynomials

$$p_k(x) = \frac{1}{x} f_k(x)$$

are of degree $n<k-1$.

Our main theorem has been surmised for some time. An attempt to prove it, by using a majorating function, which failed, is described by M. Levin [6].

To G. Szekeres [7] is due a detailed study of the structure of iteration which may yield further results connected with the problem of analyticity. This paper also includes an ample bibliography of the subject.

The authors have vainly attempted, together with G. Szekeres, to give an answer to the following question:

If $F(z)$ admits iterates in the set S but has no analytic iterate, can S be dense on the real axis (or in the complex plane)?

This question is still open. Our Main Theorem only shows that the set S is then of one-dimensional (or two-dimensional) measure 0. The only known result in the field is due to I. N. Barker [1] to the effect that, for the particular function $F(z) = e^z - 1$, the set S on the real axis reduces to the set of integers.

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