PROBLEMS AND SOLUTIONS

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Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Submitted solutions should arrive at that address before April 30, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11537. *Proposed by Lang Withers, Jr., MITRE, McClean, VA.* Let p be a prime and a be a positive integer. Let X be a random variable having a Poisson distribution with mean a, and let M be the pth moment of X. Prove that $M \equiv 2a \pmod{p}$.

11538. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Prove that a finite commutative ring in which every element can be written as a product of two (not necessarily distinct) elements has a multiplicative identity.

11539. *Proposed by William C. Jagy, MSRI, Berkeley, CA.* Let E be the set of all positive integers not divisible by 2 or 3 or by any prime q represented by the quadratic form $4u^2 + 2uv + 7v^2$. (Thus, the first few members of E are 1, 5, 11, 17, 23, and 25.) Show that $4x^2 + 2xy + 7y^2 + z^3$ is not in $\{2n^3, -2n^3, 32n^3, -32n^3\}$ for $n \in E$ and $x, y, z \in \mathbb{Z}$.

11540. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Let n be an integer greater than 1, other than 4. Let p and q be positive integers less than n and relatively prime to n. Let $a = \frac{\cos(2\pi p/n)}{\cos(2\pi q/n)}$. Show that if a^k is rational for some positive integer k, then a^k is either 1 or -1.

11541. *Proposed by Nicuşor Minculete, "Dimitrie Cantemir" University, Brasov, Romania.* Let M be a point in the interior of triangle ABC. Let R_a, R_b, and R_c be the circumradii of triangles MBC, MCA, and MAB, respectively. Let $|MA|$, $|MB|$, and $|MC|$ be the distances from M to A, B, and C. Show that

$$\frac{|MA|}{R_b + R_c} + \frac{|MB|}{R_a + R_c} + \frac{|MC|}{R_a + R_b} \le \frac{3}{2}.$$

11542. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Vicenţiu Rădulescu, Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania.* Show that for $x, y, z > 1$, and for positive

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Also solved by G. Apostolopoulos (Greece), H. E. Bell (Canada), J. Bergen, A. J. Bevelacqua, W. D. Burgess, J. Cade, N. Caro, R. A. Caruthers, R. Chapman (U. K.), J. E. Cooper III, A. Habil (Syria), A. Hays, C. Lanski, O. P. Lossers (Netherlands), A. Nakhash, D. Opitz, D. Promislow (Canada), J. Simons (U. K.), T. Smith, R. Stong, R. Tauraso (Italy), X. Wang, J. W. Ward, the Microsoft Research Problems Group, the NSA Problems Group, the Texas State University Problem Solvers Group, and the proposer.

Quadratic form plus a cube

11539 [2010, 929]. *Proposed by William C. Jagy, MSRI, Berkeley, CA.* Let E be the set of all positive integers not divisible by 2 or 3 or by any prime q represented by the quadratic form $4u^2 + 2uv + 7v^2$. (Thus, the first few members of E are $1, 5, 11, 17, 23$, and 25.) Show that $4x^2 + 2xy + 7y^2 + z^3$ is not an element of $\{2n^3, -2n^3, 32n^3, -32n^3\}$ for $n \in E$ and $x, y, z \in \mathbb{Z}$.

Solution by Robin Chapman, Exeter, UK. Let $Q_1(x, y) = x^2 + 27y^2$ and $Q_2(x, y) = 4x^2 + 2xy + 7y^2$. Both Q_1 and Q_2 are primitive positive definite integral quadratic forms with discriminant -108. By special cases of quadratic and cubic reciprocity, we have the following (Theorems 2.13 and 4.15 of [1]):

(i) a prime p is represented by one of these forms if and only if $p \equiv 1 \pmod 3$;

(ii) p is represented by the quadratic form Q_1 if and only if $p \equiv 1 \pmod 3$ and 2 is a cubic residue modulo p.

Therefore, if p is represented by Q_2, then 2 is not a cubic residue modulo p.

Now suppose that $4x^2 + 2xy + 7y^2 + z^3 = kn^3$ with $k \in \{\pm 2, \pm 32\}$, for $x, y, z \in \mathbb{Z}$ and $n \in E$. We have $Q_2(x, y) = kn^3 - z^3$, but we cannot have $x = y = 0$, since k is not the cube of a rational. Now $Q_2(x, y) = \frac{1}{4}((4x + y)^2 + 27y^2) \geq \frac{27}{4} > 6$ if $y \neq 0$, and $4x^2 + 2xy + 7y^2 = 4x^2 \geq 4$ if $y = 0$.

Consider the possible values of $Q_2(x, y)$. If $Q_2(x, y)$ is even, then y is even; let $y = 2y'$. Now $Q_2(x, y) = 4(x^2 + xy' + 7(y')^2)$. For $x^2 + xy' + 7(y')^2$ to be even, both x and y' must be even. Repeating this argument, we see that the 2-adic valuation $v_2(Q_2(x, y))$ must be even. (Here $v_2(m)$ is defined as the highest power of 2 that divides m.) Since $n \in E$, n is odd, and therefore $v_2(kn^3)$ is 1 or 5. Since $z^3 = kn^3 - Q_2(x, y)$, we have $v_2(z^3) = \min(v_2(kn^3), v_2(Q_2(x, y))) \in \{0, 1, 2, 4, 5\}$. However, $v_2(z^3)$ is a multiple of 3, so $v_2(z^3) = 0$, and hence $Q_2(x, y)$ is odd.

Now $Q_2(x, y) \equiv x^2 + 2xy + y^2 = (x + y)^2 \pmod 3$. If $3 \mid Q_2(x, y)$, then $x \equiv -y \pmod 3$, so we can write $x = 3t - y$, where $t \in \mathbb{Z}$. Now $Q_2(x, y) = 36t^2 - 18t + 9y^2$ so it is a multiple of 9, so $z^3 \equiv kn^3 \pmod 9$. However, $n \in E$ yields $3 \nmid n$, so $n \equiv \pm 1 \pmod 3$. Therefore, $n^3 \equiv \pm 1 \pmod 9$, leading to $z^3 \equiv \pm 2$ or $\pm 5 \pmod 9$, which is impossible. We conclude that $Q_2(x, y)$ is coprime to 6.

With $m = \gcd(x, y)$ we get $Q_2(x, y) = m^2 Q_2(x', y')$, where $x = mx'$, $y = my'$, and $\gcd(x', y') = 1$. Now $Q_2(x', y') \geq 4$, and we claim that $Q_2(x', y')$ has a prime factor p represented by Q_2. Let p be any prime factor of $Q_2(x', y')$. We have $4Q_2(x', y') \equiv (4x' + y')^2 + 27(y')^2 \pmod p$, and hence $p \equiv 1 \pmod 3$ by (i). Now p is represented by Q_1 or Q_2; let us suppose by Q_1 so that $p = u^2 + 27v^2$.

Define $a = ux' - vx' - 7vy'$ and $b = uy' + 4vx' + vy'$ to get

$$4a^2 + 2ab + 7b^2 = (u^2 + 27v^2)\left(4(x')^2 + 2x'y' + 7(y')^2\right) = pQ_2(x', y').$$

Since $u^2 \equiv -27v^2 \pmod p$, we can write $u \equiv \xi v \pmod p$, where $\xi^2 \equiv -27 \pmod p$ (ξ exists, since -3 is a quadratic residue modulo p). Since $(4x' + y')^2 \equiv 27(y')^2 \pmod p$, we also have $4x' + y' \equiv \pm \xi y' \pmod p$. Replacing u by $-u$ and ξ by $-\xi$, if necessary, we may assume that $u \equiv \xi v$ and $4x' \equiv -(1 + \xi)y' \pmod p$. Now

$$4a \equiv \left(-\xi(1 + \xi) + (1 + \xi) - 28\right) vy' = -(\xi^2 + 27)vy' \equiv 0 \pmod p$$

and $b \equiv \left(\xi - (1+\xi) + 1\right) vy' \equiv 0 \pmod{p}$. Thus $p \mid a$ and $p \mid b$, so

$$\frac{Q_2(x', y')}{p} = \frac{Q_2(a, b)}{p^2} = Q_2\left(\frac{a}{p}, \frac{b}{p}\right)$$

and hence $Q_2(x', y')/p$ is represented by Q_2. Iterating this argument must eventually find a prime factor of $Q_2(x', y')$ represented by Q_2.

Let the prime p divide $Q_2(x', y')$ and be represented by Q_2, so 2 is not a cubic residue modulo p. We have $kn^3 \equiv z^3 \pmod{p}$. Since also $k = \pm 2$ or $k = \pm 2^5$, we conclude that k is not a cubic residue modulo p. Hence $p \mid n$, contradicting $n \in E$. This shows that $Q_2(x, y) + z^3 \notin \{\pm 2n^3, \pm 32n^3 : n \in E\}$.

References

[1] David A. Cox, *Primes of the form $x^2 + ny^2$*, John Wiley & Sons, 1989.

Also solved by the proposer.

A Stirling sum

11545 [2011, 84]. *Proposed by Manuel Kauers, Research Institute for Symbolic Computation, Linz, Austria, and Sheng-Lan Ko, National Taiwan University, Taipei, Taiwan.* Find a closed-form expression for

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} s(n+k, k),$$

where s refers to the (signed) Stirling numbers of the first kind.

Solution I by Jim Simons, Cheltenham, U. K. The answer is $\prod_{i=1}^{n}(2i-1)$. Let $c(n,k)$ denote the unsigned Stirling number of the first kind, the number of permutations of $[n]$ with k cycles. By definition, $s(n,k) = (-1)^{n-k} c(n,k)$. Subsituting this definition into the sum and then setting $k = n - i$ transforms the sum to

$$\sum_{i=0}^{n} (-1)^i \binom{2n}{i} c(2n-i, n-i).$$

Now $\binom{2n}{i} c(2n-i, n-i)$ is the number of ways to construct a permutation of $[2n]$ with n cycles by choosing i fixed points and constructing a permutation with $n - i$ cycles on the remaining $2n - i$ elements. By inclusion-exclusion, the sum is the number of permutations of $[2n]$ having n cycles and no fixed points. Each cycle in such a permutation must be a 2-cycle. It is well known that the number of pairings of $2n$ elements is $\prod_{i=1}^{n}(2i-1)$.

Solution II by Kim McInturff, Santa Barbara, CA. We obtain the answer in the form $(2n)!/2^n n!$. It is well known that

$$\sum_{n \ge k} s(n,k) \frac{t^n}{n!} u^k = (1+t)^u,$$

Now

$$\begin{aligned} e^{-ut}(1+t)^u &= \sum_{i=0}^{\infty} (-1)^i \frac{(ut)^i}{i!} \sum_{j,k} s(j+k, k) \frac{t^{j+k}}{(j+k)!} u^k \\ &= \sum_{i,j,k} (-1)^i \binom{i+j+k}{j+k} s(j+k, k) \frac{t^{i+j+k}}{(i+j+k)!} u^{i+k}. \end{aligned}$$