① $A = \begin{bmatrix} 6 & -5 \\ 13 & -10 \end{bmatrix}$ $\qquad u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$ax^2+bx+c=0$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

a) e^{At}? $\quad \det\left[A-\lambda I\right] = \det\begin{bmatrix} 6-\lambda & -5 \\ 13 & -10-\lambda \end{bmatrix}$

$= (6-\lambda)(-10-\lambda) + \underbrace{5\cdot 13}_{65} = \lambda^2+4\lambda+5$

$x^2+2px+q=0$

$x = -p\pm\sqrt{p^2-q}$

$x = -2\pm\sqrt{4-5} = -2\pm i$

$\lambda = -2-i \qquad \begin{bmatrix} 6+2+i & -5 \\ 13 & -10+2+i \end{bmatrix} = \begin{bmatrix} 8+i & -5 \\ 13 & -8+i \end{bmatrix}$

G-J $\begin{bmatrix} 1 & -\frac{5}{8+i} \\ 0 & \underset{0}{\cancel{\frac{65}{8+i} - 8+i}} \end{bmatrix}$

$\frac{65+(-8+i)(8+i)}{8+i}$

$= \frac{65-64-1}{8+i} = 0$

$x - \frac{5}{8+i}y = 0$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{8+i}y \\ y \end{bmatrix} = y\begin{bmatrix} 5/8+i \\ 1 \end{bmatrix} = \frac{5}{8+i}y\begin{bmatrix} 1 \\ 8+i/5 \end{bmatrix}$

$V = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ 8/5 \end{bmatrix}$

$P = \begin{bmatrix} 0 & 1 \\ 1/5 & 8/5 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1/5 & 8/5 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1/5 & 8/5 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 8 & 0 & 5 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -8 & 5 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ $\quad \underbrace{\begin{matrix} -8 & 5 \\ 1 & 0 \end{matrix}}_{P^{-1}}$

$$P^{-1}AP = \begin{bmatrix}-8 & 5\\ 1 & 0\end{bmatrix}\underbrace{\begin{bmatrix}6 & -5\\ 13 & -10\end{bmatrix}\begin{bmatrix}0 & 1\\ 1/5 & 8/5\end{bmatrix}}_{\begin{bmatrix}-1 & -2\\ -2 & 11\end{bmatrix}} = \begin{bmatrix}-2 & 1\\ -1 & -2\end{bmatrix}$$

$$e^{P^{-1}AP} = e^{-2t}\begin{bmatrix}c(t) & s(t)\\ -s(t) & c(t)\end{bmatrix}$$

$$e^{A} \underset{\uparrow}{=} Pe^{P^{-1}AP}P^{-1} = \begin{bmatrix}0 & 1\\ 1/5 & 8/5\end{bmatrix}e^{-2t}\underbrace{\begin{bmatrix}c(t) & s(t)\\ -s(t) & c(t)\end{bmatrix}\begin{bmatrix}-8 & 5\\ 1 & 0\end{bmatrix}}_{\begin{bmatrix}-8c+s & 5c\\ 8s+c & -5s\end{bmatrix}}$$

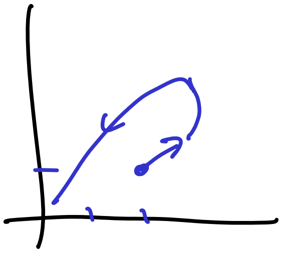
see problem 5 ☺

$$= e^{-2t}\begin{bmatrix}8\sin t+\cos t & -5\sin t\\ \underbrace{\frac{\cancel{-8c}+s}{5} + \frac{8(8s+\cancel{c})}{5}}_{13\sin t} & \cos t-8\sin t\end{bmatrix}$$

b) $x'(t) = e^{At}u =$

$$= e^{-2t}\begin{bmatrix}8\sin t+\cos t & -5\sin t\\ 13\sin t & \cos t-8\sin t\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix}$$

$$= e^{-2t}\begin{bmatrix}11\sin t+2\cos t\\ 18\sin t+\cos t\end{bmatrix}$$



② a) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} + \underbrace{\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}}_{N}$

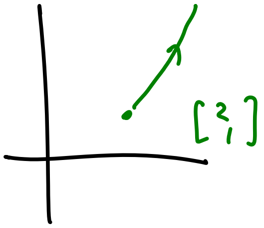
✓ Since It commutes with anything

$$e^{At} = e^{(I+N)t} = e^{It+Nt} = e^{It}e^{Nt}$$

$$= e^{t}\left(I + Nt + \underbrace{N^2\frac{t^2}{2} + \dots}_{0}\right) = e^{t}\begin{bmatrix} 1 & 0 \\ 2t & 1 \end{bmatrix}$$

b) $x'(t) = e^{At}u = e^{t}\begin{bmatrix} 1 & 0 \\ 2t & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{t} \\ (4t+1)e^{t} \end{bmatrix}$

c)



③ $A = \begin{bmatrix} 6 & -5 \\ 13 & -10 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 6 & 13 \\ -5 & -10 \end{bmatrix}\begin{bmatrix} 6 & -5 \\ 13 & -10 \end{bmatrix} = \begin{bmatrix} 205 & -160 \\ -160 & 125 \end{bmatrix}$$

charpoly : $(205-\lambda)(125-\lambda) - \underbrace{160^2}_{25600}$

$$\lambda^2 - 330\lambda + \underbrace{25625 - 25600}_{25}$$

$$\lambda = 165 \pm \sqrt{165^2 - 25} = 165 \underset{-}{\textcircled{+}} 40\sqrt{17}$$

$$\sqrt{165 + 40\sqrt{17}} \approx 18$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

charpoly: $(5-\lambda)(1-\lambda) - 4 = \lambda^2 - 6\lambda + \underbrace{5-4}_{1}$

$$\lambda = 3 \pm \sqrt{9-1} = 3 \pm \sqrt{8} = 3 \pm 2\sqrt{2}$$

$$\sqrt{3+2\sqrt{2}} \approx 2.414 = \sqrt{2}$$ ☺

④ a) $\|T\| = \max_{|\bar{x}|=1} |T\bar{x}| = \max_{\bar{x}\neq 0} \dfrac{|T\bar{x}|}{|\bar{x}|}$

Given $\bar{x} \neq 0$ write $\bar{x} = |\bar{x}|\hat{x}$

$$T(\bar{x}) = T(|\bar{x}|\hat{x}) = |\bar{x}|\, T(\hat{x})$$

$$\frac{T(\bar{x})}{|\bar{x}|} = T(\hat{x})$$

↓ Take max of both sides

b) $\forall \bar{x} \quad T(\bar{x}) \leq \|T\| \, |\bar{x}|$

If $\bar{x} = 0$, done   If not $\dfrac{|T(\bar{x})|}{|\bar{x}|} \leq \|T\|$ ☺

c) $ST(\bar{x}) = S(T\bar{x}) \leq \|S\| \, |T\bar{x}|$

$\leq \|S\| \, \|T\| \, |\bar{x}|$

$$\frac{ST(\bar{x})}{|\bar{x}|} \leq \|S\| \, \|T\|$$

max of ↗

$\therefore \|ST\| \leq \|S\| \, \|T\|$

⑤ $A \sim B \Rightarrow e^A \sim e^B$

$\uparrow$

∃ inv. matrix P $\quad AP = PB$

$$A^k P = \underbrace{A\cdot A\cdots A}_{k} P = \underbrace{A\ldots A}_{k-1} PB = \underbrace{A\ldots A}_{k-2} P\underbrace{BB}_{2} = \ldots PB^k$$

$$e^A P = \left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right)P = \sum_{k=0}^{\infty} \frac{A^k P}{k!} = \sum \frac{PB^k}{k!}$$

$$= P\left(\sum \frac{B^k}{k!}\right) = Pe^B$$

☺