Midterm 2 MAT5313 Sp.'14

① Let $z = re^{i\theta} \in \mathbb{C}\setminus\{0\}$. If z has finite order, $\exists n>0$ $z^n = 1$, i.e. $r^n e^{in\theta} = 1$ $\therefore$ $r = 1$, $\exists k$ $n\theta = 2k\pi$.
If $\gcd(k,n) = 1$, $\langle k \rangle = \mathbb{Z}_n$, so $\langle e^{i2k\pi/n}\rangle = \langle e^{i2\pi/n}\rangle$.
If not, reduce the fraction k/n.
$\therefore \langle e^{i2\pi/n}\rangle$, $n \geq 1$ are the only subgroups of finite order.

Note: $\langle e^{i2\pi/m}\rangle \vee \langle e^{i2\pi/n}\rangle = \langle e^{i2\pi/\mathrm{lcm}(m,n)}\rangle$
$\langle e^{i2\pi/m}\rangle \cap \langle e^{i2\pi/n}\rangle = \langle e^{i2\pi/\gcd(m,n)}\rangle$, so
the lattice of subgroups of finite order $\cong \mathbb{N}$ under divisibility.

② Conjugacy classes: $Z(S_4) = \{()\}$.
2-cycles: $\binom{4}{2} = 6$
3-cycles: $\binom{4}{3}\cdot 2 = 8$
4-cycles: $4!/4 = 3! = 6$
Products of 2 disjoint 2-cycles: 3

$$\boxed{4! = 24 = 1+6+8+6+3}$$

③ $\boxed{u^3} = u^2 u = (-3u-1)u = -3u^2 - u = -3(-3u-1) - u = \boxed{8u+3}$
Let $w = 1+u$, then $(w-1)^2 + 3(w-1) + 1 = 0$, so $w^2 + w - 1 = 0$, so
$w(w+1) = 1$, so $\boxed{(1+u)^{-1} = u+2}$

④ Let $w = 8u+3$, then $u = \frac{w-3}{8}$, so $\left(\frac{w-3}{8}\right)^2 + 3\left(\frac{w-3}{8}\right) + 1 = 0$,
$(w-3)^2 + 24(w-3) + 64 = 0$, $\boxed{w^2 + 18w + 1} = 0$ irrational discriminants, so ← irred. over $\mathbb{Q}$ →
Let $w = u+2$, then $u = w-2$, so $(w-2)^2 + 3(w-2) + 1 = 0$, so $\boxed{w^2 - w - 1} = 0$

⑤ Let $p(x) = x^4 - 3$. Roots: $3^{1/4} i^k$, $k = 0,1,2,3$

Galois group generators: $f, g \in \mathrm{Aut}_{\mathbb{Q}} \mathbb{Q}(3^{1/4}, i)$, where splitting field of p

$f(3^{1/4} i^k) = 3^{1/4} i^{k+1}$, $g(3^{1/4} i^k) = 3^{1/4}(-i)^k$

By inspection ord $f = 4$, ord $g = 2$. Commutation relation:

$$f^3 g\,(3^{1/4} i^k) = f^3(3^{1/4}(-i)^k) = f^3(3^{1/4} i^{3k}) = 3^{1/4} i^{3k+3}$$

$$= 3^{1/4} i^{3(k+1)} = 3^{1/4}(-i)^{k+1} = g(3^{1/4} i^{k+1}) = gf\,(3^{1/4} i^k)$$

$\therefore\ \mathrm{Aut}_{\mathbb{Q}} \mathbb{Q}(3^{1/4}, i) \cong \Delta_4$