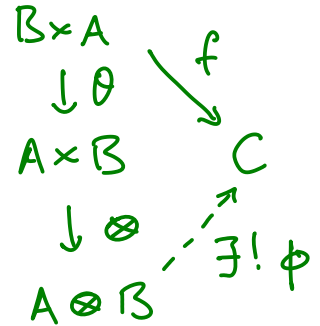


1. Let $\theta: B \times A \rightarrow A \times B$ be the bijection: $\theta(a,b) = [b,a]$
 Then $\otimes \theta: B \times A \rightarrow A \otimes B$ is bilinear:

$$\otimes \theta(kb + k'b', a) = \otimes \theta(a, kb + k'b') = k a \otimes b + k' a \otimes b' \\ = k \otimes \theta(b, a) + k' \otimes \theta(b', a), \text{ similarly in the other var. } \ddot{u}$$

If $f: B \times A \rightarrow C$ is bilinear, so is $f \theta^{-1}$.



Proof is similar to above, since it only uses bilinearity of \otimes .

By universality of $\otimes \exists! \phi \quad \phi \otimes = f \theta^{-1}$

Then $\phi \otimes \theta = f$, so $\otimes \theta$ is universal.

Since universal objects are isomorphic, $A \otimes B \cong B \otimes A \ddot{u}$

2. Let $d = \gcd(m,n)$ and $f: \mathbb{Z}_m \times \mathbb{Z}_n \rightarrow V$ bilinear.

Bézout $\Rightarrow \exists s, t \in \mathbb{Z} \quad d = sm + tn$.

$$\text{Then } d f(1,1) = sm f(1,1) + tn f(1,1) = \\ = s f(m,1) + t f(1,n) = s f(0,1) + t f(1,0) = 0 \quad (*)$$

Suppose $\phi: \mathbb{Z}_m \times \mathbb{Z}_n \rightarrow \mathbb{Z}_d$ with $\phi(x,y) = xy \text{ mod } d$.

If $x \equiv x' \text{ mod } m$, then $x \equiv x' \text{ mod } d$, so $xy \equiv x'y \text{ mod } d$.

Similarly with y 's, so ϕ is well defined and clearly bilinear.

Define $\theta: \mathbb{Z} \rightarrow V$ by $\theta(k) = k f(1,1)$. Then

θ is clearly linear. By $(*)$, $d\mathbb{Z} \subseteq \ker \theta$, so

$\exists!$ linear $\psi: \mathbb{Z}_d \rightarrow V$ with $\psi([k]_d) = k f(1,1)$

$$\psi(\phi(x,y)) = \psi([xy]_d) = xy f(1,1) = f(x,y).$$

If $\psi' \circ \phi = f$, then $\psi'(1) = \psi'(\phi(1,1)) = f(1,1) = \psi(1)$.

$\therefore \psi' = \psi$, so ψ is unique, so ϕ is universal. \ddot{u}

3. Let b_1, b_2 be a basis for F and let $f \in \text{Alt}_2 F$

Given $x_1, x_2 \in F \exists! k_{11}, k_{12}, k_{21}, k_{22} \in K$ with

$$x_1 = k_{11}b_1 + k_{12}b_2, \quad x_2 = k_{21}b_1 + k_{22}b_2$$

Since f is bilinear, $f(x_1, x_2) = k_{11}k_{21}f(b_1, b_1) + k_{11}k_{22}f(b_1, b_2) + k_{12}k_{21}f(b_2, b_1) + k_{12}k_{22}f(b_2, b_2)$

Since f is alternating, $f(b_1, b_1) = f(b_2, b_2) = 0$ and $f(b_1, b_2) = -f(b_2, b_1)$, so $f(x_1, x_2) = (k_{11}k_{22} - k_{12}k_{21})f(b_1, b_2)$

\therefore Any $f \in \text{Alt}_2 F$ is a scalar multiple of \det . $\ddot{\smile}$

4. Let $i: \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$ be the injection $x \mapsto 2x$

Apply the functor $-\otimes \mathbb{Z}_2$, then

$$(i \otimes 1)(1 \otimes 1) = 2 \otimes 1 = 2(1 \otimes 1) = 1 \otimes 2 = 1 \otimes 0 = 0$$

$\therefore i \otimes 1$ is not injective. $\ddot{\smile}$

5. a) $\mathbb{Z}^2 \otimes \mathbb{Z}^4 = (\mathbb{Z}^2)^{\otimes 4} = \mathbb{Z}^8 \quad (A \otimes K^n \cong A^n)$

b) $\mathbb{Z}^2 \otimes \mathbb{Z}_4 \cong (\mathbb{Z}_4)^2$

c) $\mathbb{Z}_2 \otimes \mathbb{Z}_4 \cong \mathbb{Z}_2$ (by 2, $\mathbb{Z}_2 \times \mathbb{Z}_4 \rightarrow \mathbb{Z}_2$ given by $[x, y] \mapsto xy$ is universal bilinear)

d) $\mathbb{Z}^3 \otimes \mathbb{Q} \cong \mathbb{Q}^3$

e) $\mathbb{Z}_3 \otimes \mathbb{Q} \cong 0 \quad (a \otimes b = a \otimes \frac{3b}{3} = 3(a \otimes \frac{b}{3}) = (3a) \otimes (\frac{b}{3}) = 0 \otimes \frac{b}{3} = 0 \quad \ddot{\smile})$