Final exam 2013.5.8 MAT 5263 Alg. Topology

1. On $\mathbb{R}^2$ define an equivalence relation

$[x,y] \sim [x',y'] \iff x-x' \in \mathbb{Z} \wedge y-y' \in \mathbb{Z}$

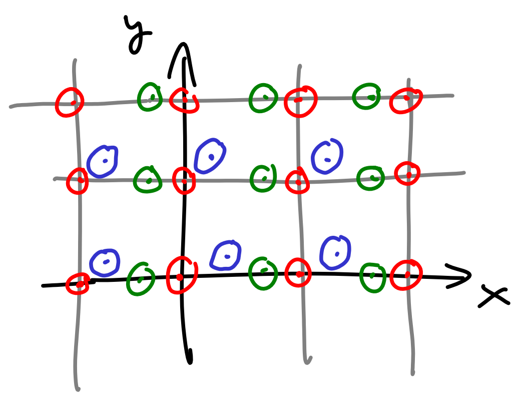
This is clearly reflexive and symmetric and easily shown to be transitive, since $x-x' \in \mathbb{Z} \wedge x'-x'' \in \mathbb{Z} \Rightarrow$ $x-x'' = x-x'+x'-x'' \in \mathbb{Z}$ and similarly for y's.

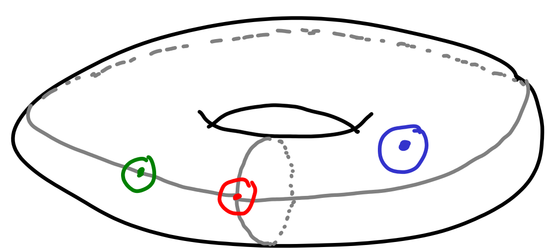
The quotient space is the same as for the unit square with opposite sides identified in the usual way to give $S^1 \times S^1$.

Let p denote the natural projection $\mathbb{R}^2 \to S^1 \times S^1$

Each point of $S^1 \times S^1$ has an evenly covered neighborhood, as illustrated below, so p is a cover. Since $\mathbb{R}^2$ is contractible, thus simply connected, this is a universal cover

$$\mathbb{R}^2 \xrightarrow{p} S^1 \times S^1$$





Deck transformations are affine shifts, leaving the integer lattice $\mathbb{Z} \times \mathbb{Z}$ invariant. Since each such shift is uniquely determined by the image of the origin, $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$

2. $$d\omega = \Big[\cancel{6x\,dx} + 6yz\sin(xz)\,dy + \cancel{\big(3y^2\sin(xz) + 3xy^2z\cos(xz)\big)dz}\Big]dx + \Big[-ayz\sin(xz)\,dx + \cancel{a\cos(xz)\,dy} + \big(-axy\sin(xz)+b\big)dz\Big]dy + \Big[\cancel{\big(3y^2\sin(xz) + 3xy^2z\cos(xz)\big)dx} + \big(6xy\sin(xz)+5\big)dy + \cancel{3x^2y^2\cos(xz)\,dz}\Big]dz$$

$$= \big[6xy\sin(xz)+5 + axy\sin(xz) - b\big]\,dy\,dz + \big[-ayz\sin(xz) - 6yz\sin(xz)\big]\,dx\,dy$$

ω is closed $\iff$ $a=-6 \;\wedge\; b=5$

$df = \omega \iff f = x^3 - 3y^2\cos(xz) + 5yz + c$

3. Since S^1 is one dimensional, all k-forms with $k>1$ are 0. By the universal property of quotient spaces, 0-forms (functions) on S^1 correspond to 2π-periodic smooth functions of θ and 1-forms (autom. closed) correspond to $\omega = F(\theta)\,d\theta$, where F is 2π-periodic.

$$\begin{tikzcd} \mathbb{R} \arrow[r] \arrow[d, "e^{i\theta}"'] & X \\ S^1 \arrow[ur] & \end{tikzcd}$$

Taking a 1-form to its integral over 1 period gives an onto map $\phi: Z^1(S^1) \to \mathbb{R}$. Meanwhile, let $f(\theta) = \int_{\theta_0}^{\theta_0+\theta}\omega$.

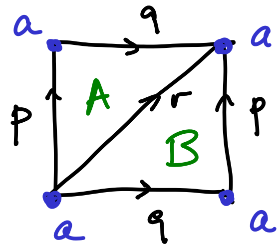
Then $df = \omega$ and f is a 0-form $\iff$ $f(\theta)$ is 2π-periodic $\iff \phi(\omega) = 0 \iff \omega \in \ker\phi$ $\quad\therefore \ker\phi = B^1(S^1)$

By the first isomorphism theorem $H^1(S^1) \cong \mathbb{R}$.

Note: $\phi(d\theta) = 2\pi$, so $d\theta$ is not exact.

Since S^1 is connected, any f on it with $df=0$ is constant, so $H^0(S^1) = \mathbb{R}$.

By vanishing thm, $H^n(S^1) = 0 \quad \forall n>1$.

4. 

$$0 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

$$\begin{array}{ccc} R^2 & R^3 & R \\ \langle A,B\rangle & \langle p,q,r\rangle & \langle a\rangle \end{array}$$

$\partial A = -p + r - q$

$\partial B = p - r + q$

$\partial p = \partial q = \partial r = a - a = 0$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad [0\ 0\ 0]$$

$\operatorname{Im} \partial_1 = 0$, so $H_0 \cong R$

$\operatorname{rk} \partial_2 = 1$, so $\ker \partial_2 = R$, so $H_0 \cong R$

Also $\ker \partial_1 = R^3$, so $H_1 \cong R^2$.

5. Parametrization:

$x = 3 \sin\phi \cos\theta \qquad -\pi \le \theta \le \pi$

$y = 3 \sin\phi \sin\theta \qquad \frac{\pi}{2} \le \phi \le \pi$

$z = 3 \cos\phi$

$\omega = y\, dy\, dz - x\, dz\, dx - z\, dx\, dy$

$d\omega = -dz\, dx\, dy = -dx\, dy\, dz$

Let Ω be the lower half-ball. Then by F.T.C.

$$\oint_{\Omega} d\omega = \oint_{\partial\Omega} \omega = \oint \omega + \underbrace{\oint \omega}_{=0}$$

$"0$ since on the disc $dz = 0$ and $z = 0$

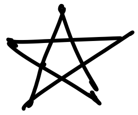
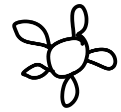
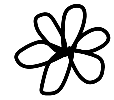
$$-\int_{\Omega} dx\,dy\,dz = -\operatorname{Vol}(\Omega) = -\frac{1}{2}\,\frac{4}{3}\pi\, 3^3 = -18\pi$$

$\therefore \oint \omega = -16\pi$

6.
$$\begin{tikzcd} \ker\phi \arrow[r, "i"] & A \arrow[r, "\phi"] & B \\ & X \arrow[ul, dotted, "\exists!\psi"] \arrow[u, "g"'] & \end{tikzcd}$$

Clearly $\phi \circ i = 0$

If $\phi \circ g = 0$, $g(X) \subseteq \ker\phi$, so for $x \in X$ define $\psi(x) = g(x)$, then $i \circ \psi = g$ and $i \circ \psi = g \Rightarrow \psi(x) = g(x)$ so we have uniqueness.

7.  is homotopically equivalent to  so to 

– a wedge of 6 pointed circles. By van Kampen's theorem

$\pi_1 = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{6}$ (free group on 6 generators)

$\therefore\ H_1 = ab(\pi_1) = \mathbb{Z}^6$ (free abelian group on 6 generators)

Since the pentagram is path connected $H_0 = \mathbb{Z}$.

By vanishing, $H_k = 0 \quad \forall k > 1$.

8. See solution to the second midterm.