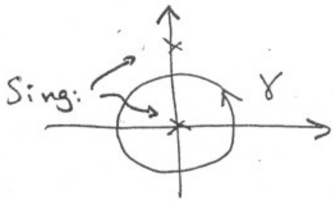
Midterm 2, MAT 5223, Fall 2002

(1) a) $z^3 - 2iz^2 = z^2(z-2i)$



$$\frac{1}{z^2(z-2i)} = \frac{1}{z^2}\left(-\frac{1}{2i}\right)\frac{1}{1-\frac{z}{2i}} = \frac{i}{2z^2}\sum_{n=0}^{\infty}\left(\frac{z}{2i}\right)^n = \frac{i}{2}\sum_{n=0}^{\infty}\frac{1}{(2i)^n}z^{n-2} =$$

$$= \frac{i}{2}\left[\frac{1}{z^2} + \frac{1}{2i}\frac{1}{z} + \frac{1}{(2i)^2} + \dots\right]$$

conv for $\left|\frac{z}{2i}\right| < 1$, i.e. $|z| < 2$

Residue $= \frac{i}{2}\cdot\frac{1}{2i} = \frac{1}{4}$

Ans: $2\pi i \cdot \frac{1}{4} = \boxed{\frac{\pi i}{2}}$

Alternate technique:

By Cauchy's integral formula $\int_\gamma \frac{f(z)\,dz}{(z-a)^2} = 2\pi i f'(a)$

$$\left[\frac{1}{z-2i}\right]' = -\frac{1}{(z-2i)^2}, \text{ so } \int_\gamma \frac{\left[\frac{1}{z-2i}\right]}{z^2}\,dz = 2\pi i\frac{-1}{(-2i)^2} = \frac{\pi i}{2}$$

b) $z = 1 - i + e^{it}$, $-\pi \le t \le \pi$, $dz = ie^{it}\,dt$, $\bar{z} = 1 + i + e^{-it}$

$$\int_\gamma \bar{z}\,dz = \int_{-\pi}^{\pi}(1+i+e^{-it})\,ie^{it}\,dt = i\int_{-\pi}^{\pi}\left[(1+i)e^{it}+1\right]dt = i(1+i)\underbrace{\frac{e^{it}}{i}\Big|_{-\pi}^{\pi}}_{0} + i2\pi = \boxed{2\pi i}$$

(2)

$$|I(r)| \le \int_\gamma \frac{1}{|z^5+1|}\,|dz| \le \int_\gamma \frac{1}{r^5-1}\,|dz|$$

← Note: $r > 1$

$$= \frac{1}{r^5-1}\int_\gamma |dz| = \frac{1}{r^5-1}\,\pi r \to 0 \quad \text{as } r \to \infty.$$

(3) Since γ is compact, $\exists M_n = \max\{|f(z)-f_n(z)| : z \in \gamma\}$ and $f_n \to f$ uniform on γ so $M_n \to 0$ as $n \to \infty$. Let $L = \int_\gamma |dz|$ be the arclength. Then

$$\left| \int_\gamma f(z)\,dz - \int_\gamma f_n(z)\,dz \right| = \left| \int_\gamma [f(z)-f_n(z)]\,dz \right| \le \int_\gamma |f(z)-f_n(z)| \cdot |dz|$$

$$= \int_\gamma M_n |dz| = M_n \int_\gamma |dz| = M_n L \to 0 \quad \text{as } n \to \infty$$

(4) Suppose $|f(z)|$ has a local min @ $z_0 \in \Omega$

Then $\exists$ disc D around z_0 such that $|f(z_0)| = \min\{|f(z)| : z \in D\}$

If $f(z_0) \neq 0$, then since $|f(z)| \ge |f(z_0)|$ we have $f(z) \neq 0$ $\forall z \in D$

Therefore $\frac{1}{f(z)}$ is $\in \mathcal{H}(D)$, but $\left|\frac{1}{f(z_0)}\right| = \max\left\{\left|\frac{1}{f(z)}\right| : z \in D\right\}$

So by the Maximum Modulus Principle $\frac{1}{f(z)}$ is constant on D

so $f(z)$ is constant on D and by the Principle of analytic continuation $f(z)$ is constant on Ω ∴ QED

(5) Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Let $\gamma = \{z : |z| = r\}$

For all sufficiently large r, on γ $|f(z)| \le |z| = r$

By Cauchy's Inequality $|c_n| \le \frac{r}{r^n} \to 0$ as $n \to \infty$ if $n > 1$

Thus, $c_n = 0$ for $n > 1$, so $f(z) = c_0 + c_1 z$