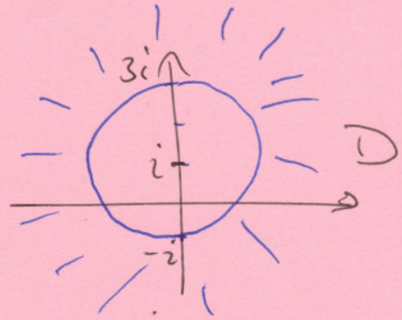
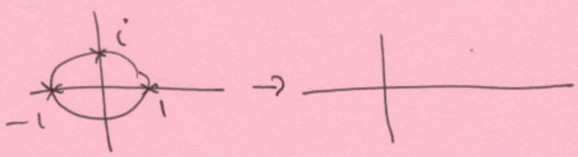
① MAT 5223 Fall 2002



First let's take the unit circle to the real axis

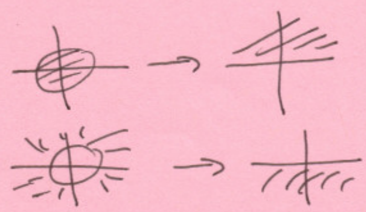


$1 \to 0$
$i \to 1$
$-1 \to \infty$

$$\frac{z-1}{z+1}\cdot\underbrace{\frac{i+1}{i-1}} = \frac{-iz+i}{z+1}$$

$$\frac{(i+1)^2}{(i-1)(i+1)} = \frac{i^2+2i+1^2}{i^2-1^2} = \frac{2i}{-2} = -i$$

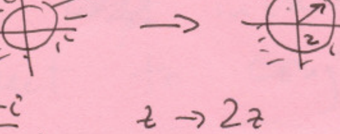
In this case $0 \mapsto i$ so



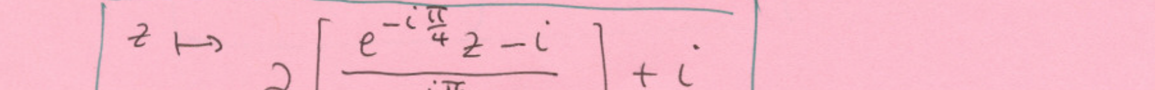
or

Now take the inverse:

$$\frac{z-i}{-z-i}$$



Now let's put everything together



$$\underbrace{z \mapsto e^{-i\frac{\pi}{4}}z \qquad z \mapsto \frac{z-i}{-z-i} \qquad z \to 2z \qquad z \mapsto z+i}$$

$$\boxed{z \mapsto 2\left[\frac{e^{-i\frac{\pi}{4}}z-i}{-e^{-i\pi/4}z-i}\right]+i}$$

Not unique. For example, I could have inserted a rotation at the unit disc stage.

(2)

Suppose $f(\underbrace{z}_{=x+iy}) = \overbrace{u(x,y) + i\,v(x,y)}^{\text{real}}$

then C-R $\Rightarrow$ $u_x = v_y$, $u_y = -v_x$.

Now let's look at g:

$$g(z) = \overline{f(\bar z)} = \overline{f(x-iy)} = u(x,-y) - i\,v(x,-y)$$

Check C-R for g:

$$[u(x,-y)]_x \stackrel{?}{=} [-v(x,-y)]_y$$

$$u_x(x,-y) \quad\cancel{=}\quad -v_y(x,-y)\cdot(-1) = v_y(x,-y)$$

chain Rule

ok

$$[u(x,-y)]_y \stackrel{?}{=} -[-v(x,-y)]_x$$

$$u_y(x,-y)(-1) = -u_y(x,-y) \qquad\qquad +v_x(x,-y)$$

ok

Thus, f satisfies C-R at z $\Leftrightarrow$ g satisfies CR at $\bar z$

The last bit of the puzzle is that D is ~~[illegible]~~ invariant with respect to conjugation, i.e. $z \in D \Leftrightarrow \bar z \in D$

$$\boxed{\text{Therefore } g \in \mathcal{H}(D)}$$

③ f is not conformal.

Let $z_1 = 1$, $z_2 = i$

the angle between z_2 & z_1 is $\frac{\pi}{2}$

whereas the angle between $f(z_2)$ & $f(z_1)$

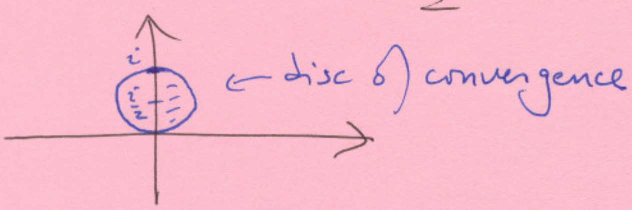
$$f(z_2) = -i, \quad f(z_1) = 1$$

is $-\frac{\pi}{2}$.

④ a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2z-i)^n}{n} = \sum_{n=1} \frac{\left[2\left(z-\frac{i}{2}\right)\right]^n}{n} = \sum_{n=1}^{\infty} \frac{2^n}{n}\left(z-\frac{i}{2}\right)^n$

↑ power series.

Ratio test: $\displaystyle\frac{2^n}{n}\frac{n+1}{2^{n+1}} = \frac{n+1}{n}\frac{1}{2} \to \frac{1}{2}$ ← radius.



b) Boundary? Depends.

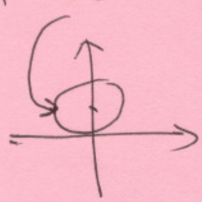
If $z = \frac{i}{2} + \frac{1}{2}$ then we get $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ ← diverges

By the integral test:

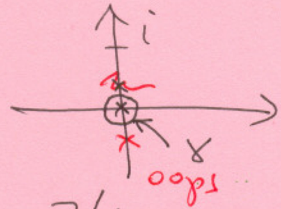
$$\int_1^{\infty} \frac{1}{x}\,dx = \ln x \Big|_1^{\infty} = \infty - 0 = \infty$$

If $z = \frac{i}{2} - \frac{1}{2}$ then we get $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ ← Converges

By the alternating series test since $\frac{1}{n} \to 0$ monotonically



⑤ a) $3z^3 + iz^2 = 3z^2\left(z + \frac{i}{3}\right)$



Using Cauchy Int. formula $\Rightarrow \displaystyle\int_\gamma \frac{dz}{3z^3+iz^2} = \frac{1}{3}\int_\gamma \frac{\left(\frac{1}{z+\frac{i}{3}}\right)dz}{z^2} = \frac{2\pi i}{3}\left[\frac{1}{z+\frac{i}{3}}\right]'\Bigg|_{z=0}$

$$= \frac{2\pi i}{3}\left(-\frac{1}{\left(z+\frac{i}{3}\right)^2}\right)\Bigg|_{z=0} = \frac{2\pi i}{3}\,\frac{-1}{-1/3^2} = \frac{2\pi i}{3}\cdot 9 = 6\pi i$$

Using series: $\displaystyle\frac{1}{3z^3+iz^2} = \frac{1}{iz^2\left(1+\frac{3}{i}z\right)} = \frac{1}{iz^2}\,\frac{1}{1-3iz} = \frac{1}{iz^2}\left(1+3iz+\ldots\right)$

$$= \frac{1}{iz^2} + \frac{3}{z} \ldots \qquad \text{Residue} = 3 \qquad \text{Ans: } 2\pi i \cdot 3 = 6\pi i$$

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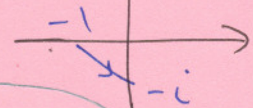
b) $z = -1 + t\left[-i-(-1)\right] = -1 + t(1-i)$

$dz = (1-i)\,dt \qquad \mathrm{Im}(z) = -t$



$$\int_\gamma \mathrm{Im}(z)\,dz = \int_0^1 [-t](1-i)\,dt = (i-1)\int_0^1 t\,dt = (i-1)\frac{1}{2}$$

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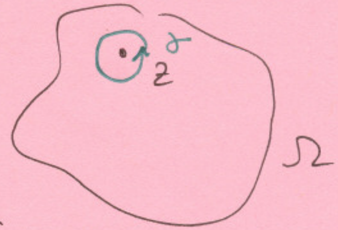
⑥ $\displaystyle\left|\int_0^1 t^{\frac{1}{2}} e^{it}\,dt\right| \le \int_0^1 \left|t^{\frac{1}{2}} e^{it}\right|\,|dt|$

$$= \int_0^1 t^{\frac{1}{2}}\,dt = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_0^1 = \frac{2}{3}t^{3/2}\Big|_0^1 = \frac{2}{3}$$

(7) Let $z \in \Omega$



Since Ω is open we may find a circle γ around z within Ω.

By Cauchy Int. Formula

$$f_n'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(\xi)}{(\xi - z)^2} \, d\xi$$

Since γ is compact $f_n \to f$ unif. on γ so we may take limit under the integral sign

$$\lim_{n\to\infty} f_n'(z) = \frac{1}{2\pi i} \int_\gamma \frac{\lim_{n\to\infty} f_n(\xi)}{(\xi - z)^2} \, d\xi$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^2} \, d\xi = f'(z)$$

(8) a) Since $f \in \mathcal{H}(D)$, $f(z) = \sum_{n=0}^{\infty} c_n z^n$ conv. on D

Since $c_0 = f(0) = 0$, $f(z) = \sum_{n=1}^{\infty} c_n z^n = z \sum_{n=1}^{\infty} c_n z^{n-1} = z \underset{\text{shift index.}}{\sum_{n=0}^{\infty}} c_{n+1} z^n$.

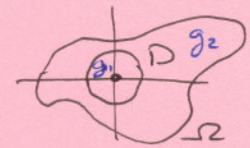
$\boxed{g_1(0) = c_1 = f'(0)}$

Let $g_1(z) = \sum_{n=0}^{\infty} c_{n+1} z^n$ on D

Let $g_2(z) = \dfrac{f(z)}{z}$ on $\Omega \setminus \{0\}$

$\Big\}$ Both analytic.

Then g_1 & g_2 agree on $\underbrace{D \setminus \{0\}}_{\text{has limit points}}$



So By the principle of analytic continuation

$\exists\, g \in \mathcal{H}(\Omega)$ s.t. g agrees with g_1 & g_2

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b) Since $\bar{D}$ is <u>compact</u> & $|f|$ & $|g|$ are <u>continuous</u>, their maxima exist on $\bar{D}$.

By the maximum modulus principle the maxima occur on ∂D, where $|z| = 1$

But then $|f(z)| = |z g(z)| = \underbrace{|z|}_{1} |g(z)| = |g(z)|$