1 a) $1 \cdot 1 = 1$, so $1 \in U(R)$

Let $u, v \in U(R)$, then $uv^{-1}vu^{-1} = 1$

$\therefore$ $uv^{-1} \in U(R)$

b) "$\Rightarrow$" $0 \notin U(R)$, $0 \in R \setminus U(R)$

Let $x, y \in R \setminus U(R)$, then $\langle x \rangle, \langle y \rangle$ are proper ideals.

If $\langle x \rangle = R$, then $\exists r \in R$ $rx = 1$, so $x \in U(R)$

By Zorn's lemma $\exists$ max ideals $M_x \supseteq \langle x \rangle$, $M_y \supseteq \langle y \rangle$.

Since R is local $M_x = M_y$, so $x - y \in M_x = M_y$, so $x-y \notin U(R)$

Let $x \in R \setminus U(R)$, $r \in R$. If xr were a unit, then $\exists u$ st. $xru = 1$, but then x is a unit

"$\Leftarrow$" Let I be a proper ideal. Then $I \cap U(R) = \emptyset$.

So $I \subseteq R \setminus U(R)$. $\therefore$ $R \setminus U(R)$ is the only max. ideal.

② a) $H = \{g \in G : mg = 0\}$

$m \cdot 0 = 0 \quad \therefore 0 \in H$

Let $x, y \in H$, then $mx = 0$, $my = 0$, so

$m(x-y) = mx - my = 0 - 0 = 0$, so $x - y \in H$

b) $\mathbb{Z}$ is free on $\{1\}$, so

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d, hook] & G \\ \mathbb{Z} \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

$\phi : \mathrm{Hom}(\mathbb{Z}, G) \to G$

$f \longmapsto f(1)$ is a 1-1 corresp.

$\phi(f + f') = (f + f')(1) = f(1) + f'(1) = \phi(f) + \phi(f')$

$\therefore \phi$ is an iso.

c) Let $\pi : \mathbb{Z} \to \mathbb{Z}_m$ be the canonical projection $(\pi(k) = [k]_m)$

Let $\psi : \mathrm{Hom}(\mathbb{Z}_m, G) \to \mathrm{Hom}(\mathbb{Z}, G)$ be the image of π under the left hom functor $\mathrm{Hom}[-, G]$, (contravariant)

i.e. $\psi(h) = h \circ \pi$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\pi"] \arrow[d, dashed] & \mathbb{Z}_m \arrow[dl, "h"] \\ G & \end{tikzcd}$$

By the universal property of quotients ψ is 1-1.

$\psi_*(\mathrm{Hom}(\mathbb{Z}_m, G)) = \{f \in \mathrm{Hom}(\mathbb{Z}, G) : \underbrace{f_*(\underset{\langle m \rangle}{m\mathbb{Z}}) = 0}_{\substack{f(m)=0 \\ \| \\ m f(1)}}\}$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "f"'] & \mathbb{Z}_m \arrow[dl, dashed, "\exists!\, h"] \\ G & \end{tikzcd}$$

with $f = h\pi$

Since ϕ, ψ are 1-1,

So is $\phi \circ \psi$

$(\phi \circ \psi)_*(\mathrm{Hom}(\mathbb{Z}_m, G)) = \phi_*(\psi_*(\mathrm{Hom}(\mathbb{Z}_m, G)))$

$\phi_*(\{f \in \mathrm{Hom}(\mathbb{Z}, G) : m f(1) = 0\}) = \{\phi(f) : m f(1) = 0\}$

$= \{f(1) : m f(1) = 0\} = \{g : mg = 0\} = H. \quad \therefore \mathrm{Hom}(\mathbb{Z}_m, G) \cong H$

(3) Let $d = \gcd(m,n)$ and $m = sd$, $n = td$

Claim: $H = \langle t \rangle$

$mt = sdt = sn \equiv 0 \bmod n$ $\therefore$ $t \in H$

Conversely let $g \in H$, then $mg \equiv 0 \bmod n$,

so $\exists k$ $mg = kn$, so $sdg = ktd$, so $sg = kt$

Since $t \mid sg$ and $\gcd(t,s) = 1$, $t \mid g$

Since $dt = n \equiv 0 \bmod n$ and d is smallest such,

$|t| = d$, so $|H| = d = \gcd(m,n)$

(4)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"'] & R \\ F \arrow[ur, dashed] & \end{tikzcd}$$

$\exists!\, \phi$ with $f = \phi \circ i$

$F \leftarrow R^{(X)}$

Since F is free, we have a 1-1 corresp $\theta : F^* \to R^X$

$\phi \mapsto f$

Given $\phi, \psi \in F^*$, let $x \in X$

$$[\theta(\phi + \psi)](x) = (\phi + \psi)(x) = \phi(x) + \psi(x) =$$

$$= \theta(\phi)(x) + \theta(\psi)(x) = [\theta(\phi) + \theta(\psi)](x)$$

$\therefore$ $\theta(\phi + \psi) = \theta(\phi) + \theta(\psi)$ $\therefore$ θ is an iso.

(5) Let $A \neq \emptyset$ (indexing set), F_α - free R-mod on X_α $(\alpha \in A)$

<u>Notation:</u> $k_\alpha : X_\alpha \hookrightarrow F_\alpha$

<u>Claim</u> $\coprod_{\alpha \in A} F_\alpha$ is free on $\coprod_{\alpha \in A} X_\alpha$

$$F_\beta \xhookrightarrow{j_\beta} \bigoplus_{\alpha \in A} F_\alpha \qquad X_\beta \xhookrightarrow{i_\beta} \biguplus_{\alpha \in A} X_\alpha$$

$$z \longmapsto \left(\gamma \mapsto \begin{cases} z & \text{if } \gamma = \beta \\ 0 & \text{otherwise} \end{cases} \right)$$

$$\begin{tikzcd}
X_\beta \arrow[r, hook, "i_\beta"] \arrow[d, hook, "k_\beta"'] & \biguplus X_\alpha \arrow[d, dashed, "\exists ! k"] \arrow[dr, bend left, "f"] & \\
F_\beta \arrow[r, hook] \arrow[rr, dashed, bend right, "j_\beta", "\varphi_\beta"'] & \bigoplus F_\alpha \arrow[r, dotted, "\varphi"] & M
\end{tikzcd}$$

By the univ. prop. of $\coprod X_\alpha$
$\exists ! k : \biguplus X_\alpha \to \bigoplus F_\alpha$ s.t.

$$k i_\beta = j_\beta k_\beta$$

Claim: k is 1-1

Let $x \in X_\alpha$, $x' \in X_\beta$ s.t. $\underbrace{k i_\alpha x}_{j_\alpha k_\alpha x} = \underbrace{k i_\beta x'}_{j_\beta k_\beta x'}$

$$\gamma \mapsto \begin{cases} k_\alpha x & \text{if } \gamma = \alpha \\ 0 & \text{oth.} \end{cases} \qquad \gamma \mapsto \begin{cases} k_\beta x' & \text{if } \gamma = \beta \\ 0 & \text{oth.} \end{cases}$$

If $\alpha = \beta$, $k_\alpha x = k_\alpha x'$, $x = x'$

If $\alpha \neq \beta$, $\forall \gamma$ $j_\alpha k_\alpha x = 0$ or $j_\beta k_\beta x' = 0$,

so $x = 0$, $x' = 0$

Since F_β is free on X_β, $\exists!\ \phi_\beta : F_\beta \to M$ s.t. $f \circ i_\beta = \phi_\beta k_\beta$

By the univ. property of $\coprod F_\alpha$, $\exists!\ \phi$ s.t. $\phi_\beta = \phi \circ j_\beta$

<u>Claim:</u> $f = \phi k$

Let $x \in \dot{\bigcup} X_\alpha$. $\exists \beta \in A$ $x = i_\beta(x_\beta)$ for some $x_\beta \in X_\beta$

$$f(x) = f i_\beta x_\beta = \phi_\beta k_\beta x_\beta = \phi j_\beta k_\beta x_\beta$$

$$= \phi k i_\beta x_\beta = \phi k(x)$$