① G - group $G \xrightarrow{\pi} \frac{G}{[G,G]}$ is universal among homs from G to abelian groups.

a) $\frac{G}{[G,G]}$ is Abelian

Note: In general $xy = yx \iff xyx^{-1}y^{-1} = e$

Let $x[G,G],\ y[G,G] \in \frac{G}{[G,G]}$

Then $x[G,G]\,y[G,G]\,(x[G,G])^{-1}(y[G,G])^{-1}$

$= \underbrace{xyx^{-1}y^{-1}}_{\in [G,G]}[G,G] = [G,G]$

b) Let A be an Abelian group and $G \xrightarrow{f} A$ is a group hom.

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "f"'] & \frac{G}{[G,G]} \arrow[dl, dashed, "\exists!\,\phi"] \\ A & \end{tikzcd}$$

$\phi \circ \pi = f$

Claim: $[G,G] \subseteq \ker f$

Let $\prod_{i=1}^{n} g_i h_i g_i^{-1} h_i^{-1} \in [G,G]$

$f\left(\prod_{i=1}^{n} g_i h_i g_i^{-1} h_i^{-1}\right) = \prod_{i=1}^{n} f(g_i) f(h_i) f(g_i)^{-1} f(h_i)^{-1}$

$= e$ (since A is Abelian)

$\therefore$ By the universal property of quotient groups $\exists!\ \phi : \frac{G}{[G,G]} \to A$ s.t. $\phi \circ \pi = f$

Alt. Define $\phi : \frac{G}{[G,G]} \to A$ by

$\phi(x[G,G]) = f(x)$ Well-defined?

Suppose $x \sim x'$, then $x^{-1}x' \in [G,G]$

$f(x^{-1}x') = e$ (see above)

$\overset{\shortparallel}{f(x)^{-1} f(x')}$ $\therefore f(x') = f(x)$ ☺

(2) If $G, H \trianglelefteq K$, $G \cap H = \{e\}$, $G \vee H = K$, then $K \cong G \times H$

Claim: $g \in G$, $h \in H \Rightarrow gh = hg$

$$\left.\begin{array}{l} \overbrace{h g^{-1} h}^{\in G \text{ (since } G \trianglelefteq K)} \in G \\ \underbrace{g h g^{-1}}_{\in H \text{ (since } H \trianglelefteq K)} h \in H \end{array}\right\} ghg^{-1}h \in G \cap H = \{e\}$$ ☺

Define $\theta : G \times H \to K$
$[g, h] \mapsto gh$

(i) θ is a hom: $\theta([g,h] \cdot [g',h'])$

$= \theta([gg', hh']) = gg'hh' = ghg'h'$

$= \theta([g,h]) \cdot \theta([g',h'])$ ☺

(ii) θ is injective.

Suppose $\theta[g,h] = e$
$\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{gh}$

$\therefore\ \underset{\in G}{g} = \underset{\in H}{h^{-1}} \in G \cap H = \{e\}$

$g = h^{-1} = e$, so $h = e$ ☺

(iii) θ is surjective: Let $k \in K = G \vee H$

Then $k = \prod_{i=1}^{n} g_i h_i = g_1 h_1 \cdots g_n h_n = \underbrace{\prod_{i=1}^{n} g_i}_{\in G} \cdot \underbrace{\prod_{i=1}^{n} h_i}_{\in H}$

$= \theta\left(\left[\prod_{i=1}^{n} g_i, \prod_{i=1}^{n} h_i\right]\right)$ ☺

(3) $\mathrm{End}(\mathbb{Q}[x]),\ \mathrm{Aut}(\mathbb{Q}[x]) = ?$

Let $f \in \mathrm{End}(\mathbb{Q}[x])$.

__Claim__ f keeps $\mathbb{Q}$ fixed (i.e. f is a $\mathbb{Q}$-algebra hom)

$f(0) = 0$ ✓

$f(1) = 1$ ✓

$f(2) = f(1+1) = f(1) + f(1) = 1 + 1 = 2$

etc.

$f(n) = f(n-1+1) = \underbrace{f(n-1)}_{\text{By Induction} = n-1} + f(1) = n - 1 + 1 = n$ ☺

$\therefore\ f$ keeps $\{0, 1, 2 \ldots\}$ fixed.

Suppose $n \in \{\dots,-2,-1\}$, then $n = -k$, $k \in \mathbb{N}$

$0 = -k + k \qquad 0 = f(0) = f(-k+k) = f(-k) + f(k)$

So solve:

$$f(n) = f(-k) = -f(k) = -k = n$$

$\therefore$ f fixes $\mathbb{Z}$

Let $\frac{m}{n} \in \mathbb{Q}$, $n \neq 0$ $\qquad \frac{m}{n} \cdot n = m$

$$f\left(\frac{m}{n} \cdot n\right) = f(m) = m$$

$$\overset{\shortparallel}{f\left(\frac{m}{n}\right) \cdot \underbrace{f(n)}_{n}}$$

Solve: $f\left(\frac{m}{n}\right) = \frac{m}{n}$ ☺

$$\begin{tikzcd}
\{x\} \arrow[r, "f"] \arrow[d, "i"'] & \mathbb{Q}[x] \\
\mathbb{Q}[x] \arrow[ur, dashed, "\exists!\,\phi"']
\end{tikzcd}
\qquad \phi \circ i = f$$

By Univ. property of free $\mathbb{Q}$-algebras.

$\therefore$ Any endo $\mathbb{Q}[x] \circlearrowleft$ is uniquely det. By where x goes.

more explicitly

Let $g \in \mathbb{Q}[x]$, $g = a_0 + a_1 x + \dots + a_n x^n$

$$f(g) = f(a_0 + \dots + a_n x^n) = \underbrace{f(a_0)}_{a_0} + \dots + \underbrace{f(a_n)}_{a_n} f(x)^n$$

$$= E_{f}(g) \quad ☺$$

Suppose $E_f \in \mathrm{Aut}(\mathbb{Q}[x])$

If $\deg f = 0$, then $f = \text{const.}$, so $E_f(g) = \text{const}$ for any g, so E_f is not surjective

$(\text{Image } E_f = \mathbb{Q})$

If $\deg f \geq 2$, then for any g, $E_f(g) \neq x$

$$\left(\begin{array}{l} \text{If } g = \text{const.}, \ E_f(g) = \text{const.} \\ \text{If } g \neq \text{const.}, \ \deg E_f(g) = \underbrace{\deg f}_{\geq 2} \cdot \underbrace{\deg g}_{>0} \geq 2 \end{array}\right)$$

$\therefore \deg f = 1$

Suppose $\deg f = 1$, then $f = a_0 + a_1 x$, $a_1 \neq 0$

and $E_f^{-1} = E_{a_1^{-1}x - a_1^{-1}a_0}$

$a_1^{-1}(a_0 + a_1 x) - a_1^{-1}a_0$
$= a_1^{-1}a_0 + x - a_1^{-1}a_0 = x$ ☺

$\therefore \mathrm{Aut}(\mathbb{Q}[x]) = \{ E_f : \deg f = 1 \}$

④ R ring, $nR = \{nr : r \in R\}$

$nr = \underbrace{r + \dots + r}_{n}$

a) $0 = n0 \in nR$

If $nr, nr' \in nR$

$nr - nr' = n(r - r') \in nR$

$\therefore nR$ is an additive subgroup.

Let $x \in R$, $nr \in nR$

$x(nr) = x(\underbrace{r + \dots + r}_{n}) = \underbrace{xr + \dots + xr}_{n} = n(xr) \in nR$

$(nr)x = (r + \dots + r)x = rx + \dots + rx = n(rx) \in nR$

$\therefore nR$ is an ideal.

b) Suppose $R \xrightarrow{f} R'$ is a ring hom.

$$\begin{tikzcd} R \arrow[r, "\pi"] \arrow[d, "f"'] \arrow[dr] & \frac{R}{nR} \arrow[d, dotted, "\exists!\phi"] \\ R' \arrow[r, "\pi'"'] & \frac{R'}{nR} \end{tikzcd}$$

Claim $nR \subseteq \ker(\pi' \circ f)$

Let $nr \in R$, $\pi'(f(nr))$

$= \pi'(\underbrace{n f(r)}_{\in nR'}) = 0$ ☺

By the universal property of quotients, $\exists! \phi$ s.t. $\phi \circ \pi = \pi' \circ f$

Define $\mathcal{F}$: Rings ↺

On objects $\mathcal{F}(R) = \frac{R}{nR}$

on morphisms $\mathcal{F}(f) = \phi$ ← unique! (as above) c)

$$\begin{tikzcd}
R \arrow[r, "\pi"] \arrow[d, "f"'] & \frac{R}{nR} \arrow[d, dashed, "\mathcal{F}(f)"] \arrow[dd, bend left=60, dashed, "\mathcal{F}(f'\circ f)"] \\
R' \arrow[r, "\pi'"] \arrow[d, "f'"'] & \frac{R'}{nR'} \arrow[d, dashed, "\mathcal{F}(f')"] \\
R'' \arrow[r, "\pi''"] & \frac{R''}{nR''}
\end{tikzcd}$$

$\mathcal{F}(f)\circ\pi = \pi'\circ f$

$\mathcal{F}(f')\circ\pi' = \pi''\circ f'$

$\mathcal{F}(f'\circ f)\circ\pi = \pi''\circ f'\circ f$

$\mathcal{F}(f')\circ\mathcal{F}(f)\circ\pi = \mathcal{F}(f')\circ\pi'\circ f = \pi''\circ f'\circ f$

By uniqueness $\mathcal{F}(f'\circ f) = \mathcal{F}(f')\circ\mathcal{F}(f)$

$\therefore$ $\mathcal{F}$ is a functor

⑤ Suppose $\langle x-c\rangle \underset{\neq}{\subset} J \subset k[x]$

Let $p \in J \setminus \langle x-c\rangle$

div. alg: $\exists !\, q, r \quad p = q(x-c) + r$, $\quad \begin{matrix} r=0 \text{ or} \\ \deg r < 1 \end{matrix}$

Since $p \notin \langle x-c\rangle$, r is a nonzero const. (unit)

$r = \underbrace{p}_{\in J} - \underbrace{q(x-c)}_{\in \langle x-c\rangle \subset J} \quad \in J \qquad \therefore J = k[x]$

☺

Alt. Since $k[x]$ is a p.i.d., we can assume $J = \langle s\rangle$ for some $s \in k[x]$

Then $\exists q \quad x-c = s\cdot q$

If $s =$ const, $\langle s\rangle = k[x]$, done

otherwise $\deg s = 1$, and $q =$ const., so

s & $x-c$ are associates, so

$\langle s\rangle = \langle x-c\rangle$ ☹

Alt′ Let $\varepsilon : k[x] \to k$

$p(x) \mapsto p(c)$

ε is clearly onto: $\varepsilon(k) = k$.

Claim: $\ker \varepsilon = \langle x-c\rangle$

$\varepsilon\big((x-c)q\big) = 0$

Conversely suppose $\varepsilon(p) = 0$
$= p(c)$

Div. alg: $p = q(x-c) + r$ $\quad r = 0$ or $\deg r < 1$ $(\therefore r = \text{const})$

Plug in c: $\underset{\overset{\shortparallel}{0}}{p(c)} = 0 + r$ $\quad \therefore r = 0$

$p \in \langle x - c \rangle$ ☺

By 1^{st} iso. theorem: $\dfrac{k[x]}{\ker \varepsilon} \cong \text{Image } \varepsilon$

$$\frac{k[x]}{\langle x-c \rangle} \cong k$$

field

$\therefore$ maximal ☺