1. Prove that the special linear group $\mathrm{SL}_n(\mathbf{R})$ of all matrices with determinant 1 is a normal subgroup of the general linear group $\mathrm{GL}_n(\mathbf{R})$ of all invertible $n \times n$ matrices with real coefficients. What is the quotient group?

Pf 1 det is a hom : $GL_n(\mathbb{R}) \to \mathbb{R}^*$

$SL_n(\mathbb{R}) = \ker(\det)$

$\therefore SL_n(\mathbb{R}) \lhd GL_n(\mathbb{R})$

1st iso thm $\Rightarrow$ $\dfrac{GL_n(\mathbb{R})}{SL_n(\mathbb{R})} = \dfrac{GL_n(\mathbb{R})}{\ker(\det)} \cong \operatorname{im}(\det) = \mathbb{R}^*$

:)

Pf2 $\det(Id) = 1 \quad \therefore Id \in SL_n(\mathbb{R})$

If $A, B \in SL_n(\mathbb{R})$, $\det A = \det B = 1$, so

$\det(AB^{-1}) = \dfrac{\det A}{\det B} = \dfrac{1}{1} = 1 \quad \therefore AB^{-1} \in SL_n(\mathbb{R})$

$\therefore SL_n(\mathbb{R}) < GL_n(\mathbb{R})$

If $A \in SL_n(\mathbb{R})$, $B \in GL_n(\mathbb{R})$, then

$\det(B^{-1}AB) = \dfrac{1}{\cancel{\det B}} \cdot \det A \cdot \cancel{\det B} = 1$

$\therefore SL_n(\mathbb{R}) \lhd GL_n(\mathbb{R})$

A & B are in the same coset of $SL_n(\mathbb{R})$

$\Leftrightarrow AB^{-1} \in SL_n(\mathbb{R}) \Leftrightarrow \det(AB^{-1}) = 1$

$\Leftrightarrow \dfrac{\det A}{\det B} = 1 \Leftrightarrow \det A = \det B \quad \therefore \dfrac{GL_n(\mathbb{R})}{SL_n(\mathbb{R})} = \mathbb{R}^*$

:)

2. Prove that the set of all rotations in the dihedral group D_n of all symmetries of the regular polygon with n vertices is a normal subgroup. What is the quotient group?

Again we have a hom $\det : D_n \to \mathbb{R}^*$

Then $\ker(\det) = \underbrace{\{\text{all rotations in } D_n\}}_{H}$ $\therefore H \triangleleft D_n$

By 1^{st} iso. thm $\dfrac{D_n}{H} = \dfrac{D_n}{\ker(\det)} \cong \operatorname{im}(\det) = \{1,-1\} \cong \mathbb{Z}_2$

Long slog: $R_0 \in H$ ☺

composition of rotation is a rotation

inverse(rot) is a rotation (inverses are automatic anyway since D_n is finite)

$\therefore H < D_n$

If $R \in H$, $T \in D_n$, then

$$T^{-1}RT = \begin{cases} \text{a rotation} & \text{if } T \text{ is a rotation} \\ \text{a rotation} & \text{if } T \text{ is a flip (orientation argument)} \end{cases}$$

$= \text{a rotation} \in H$

$\therefore H \triangleleft D_n$

(Alt. reason: $|H| = n = \frac{1}{2}|D_n|$ so there is only one nontrivial coset of H : $D_n \setminus H$ {all flips})

so left & right cosets are the same $\therefore H \triangleleft D_n$

2 cosets $\Rightarrow$ quotient has 2 elements, so $\cong \mathbb{Z}_2$

3. Suppose X is a set and F is a field. Let R be the ring of all functions $X \to F$ with pointwise operations.

   (a) What are the units of R? Prove your assertion.

   (b) Use an explicit example to show that R may have zero divisors.

(a) $\mathcal{U}(R) = \{ f \in R : \forall s \in X \quad f(s) \neq 0 \}$

Pf "$\Leftarrow$" If $\forall s \in X$ $f(s) \neq 0$, define g by

$g(s) = f(s)^{-1}$ (F is a field) Then $f \cdot g = 1$, so $f \in \mathcal{U}(R)$

"$\Rightarrow$" If $f \in \mathcal{U}(R)$, $\exists g \in R$ $fg = 1$

Then $\forall s \in S$ $(fg)(s) = f(s)g(s) = 1$

$\therefore$ Since F is a field $f(s) \neq 0$ ☺

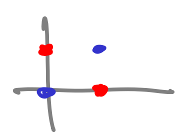
(b) Let $X = \{s, t\}$ $(s \neq t)$, $F = \mathbb{Z}_2$

Define $f, g \in R$ by

x	$f(x)$
s	0
t	1

x	$g(x)$
s	1
t	0

Then $f, g \neq 0$, But $fg \equiv 0$.



4. With R as in the preceding problem and $s \in X$, let $I = \{f \in R\colon f(s) = 0\}$. Prove that I is a maximal ideal of R.

zero function $(s) = 0 \quad \therefore \quad 0 \in I$

If $f, g \in I$, i.e. $f(s) = g(s) = 0$, $(f-g)(s) = f(s) - g(s) =$
$= 0 - 0 = 0. \quad \therefore f - g \in I \qquad \therefore I < R$

If $f \in I$, $g \in R$, $(fg)(s) = f(s)g(s) = 0 \cdot g(s) = 0$
$\therefore \quad fg \in I \qquad \therefore I$ is an ideal

Suppose J is an ideal of R with $I \subsetneq J$

Then $\exists f \in J \setminus I$. Since $f \notin I$, $f(s) \neq 0$

Let $h \in R$ defined by $h(x) = f(s) \quad \forall x \in X$ (h is const)

Since $(h-f)(s) = h(s) - f(s) = f(s) - f(s) = 0$, $h - f \in I \subset J$

$$\underset{\text{unit}}{h} = \underbrace{h - f}_{\in J} + \underset{\in J}{f} \quad \in J \qquad \therefore J = R$$

$\therefore I$ is a max ideal

Alt. proof: We have the evaluation $\boxed{\text{hom}}$ $\quad \mathcal{E} : R \to F$
$f \mapsto f(s)$

$I = \ker \mathcal{E}$. By the 1st iso. thm

$\mathcal{E}(fg) = (fg)(s) = f(s)g(s) = \mathcal{E}(f)\,\mathcal{E}(g)$
$\mathcal{E}(f+g) = (f+g)(s) = f(s) + g(s) = \mathcal{E}(f) + \mathcal{E}(g)$

$$\frac{R}{I} = \frac{R}{\ker \mathcal{E}} = \operatorname{im}(\mathcal{E}) = F \leftarrow \text{field}$$

$\therefore I$ is maximal

:)