1. Suppose X is a set, $s \in X$ and F is a field. Let R be the ring of all functions $X \to F$ with pointwise operations. Let $I = \{f \in R : f(s) = 0\}$. Prove that I is a maximal ideal of R.

Claim: I is an ideal

(i) $0 \in I$ $\quad \left(0\big|_{x=0} = 0 \right)$

(ii) Let $f, g \in I$ then $f(s) = g(s) = 0$

$\therefore (f-g)(s) = f(s) - g(s) = 0 - 0 = 0 \quad \therefore f - g \in I$

(iii) Let $f \in I$, $g \in R$. Then $f(s) = 0$

$(fg)(s) = f(s)g(s) = 0 \cdot g(s) = 0 \quad \therefore fg \in I$

$\therefore I$ is an ideal ☺

Let K be an ideal of R s.t. $I \subsetneq K$

Let $g \in K \setminus I$. Since $g \notin I$ $\quad g(s) \neq 0$

Define $h \in R$ by $h(x) = \begin{cases} 0 & \text{if } x = s \\ 1 - g(x) & \text{otherwise} \end{cases}$

Since $g \in K$, $h \in J \subset K$, $g + h \in K$

But $(g+h)(x) = \left\{ \begin{array}{ll} g(s) & \text{for } x = s \\ \underbrace{g(x) + 1 - g(x)}_{1} & \text{otherwise} \end{array} \right\} \neq 0$

$\therefore g+h$ is a unit in R

$\therefore K = R$

$\therefore I$ is maximal

With homs $\quad$ Let $\varepsilon : R \to F$ be the evaluation hom

$\varepsilon(f) = f(s)$

Then clearly ε is a surjective hom.

Also $\ker \varepsilon = I$

By 1st iso. thm $\quad \dfrac{R}{I} \cong F$

Since F is a field, I is a max. ideal of R

2. Suppose R is as in the preceding problem and $t \in X$. Let $J = \{f \in R\colon f(s) = f(t) = 0\}$. Prove that if $t \neq s$, then J is an ideal of R which is not prime.

Given $c \in X$ define $I_c = \{f \in R : f(c) = 0\}$

By #1 I_c is an ideal of R

$\therefore$ $J = I_s \cap I_t$ is an ideal of R

Given $c \in X$, define $\delta_c(x) = \begin{cases} 1 & \text{if } x = c \\ 0 & \text{otherwise} \end{cases}$

Then $\delta_s, \delta_t \notin J$

If $s \neq t$ $\delta_s \delta_t = 0 \in J$ $\therefore$ J is not a prime ideal

<u>With homs:</u> Define $\varepsilon : R \to F^2$ by

$\varepsilon(f) = [f(s), f(t)]$

$$\begin{tikzcd} & R \arrow[dl, "\varepsilon_s"'] \arrow[d, dotted, "\varepsilon"] \arrow[dr, "\varepsilon_t"] & \\ F & F \times F \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & F \end{tikzcd}$$

Then $\ker \varepsilon = J$ and

if $s \neq t$, ε is onto

Given $[u, v] \in F^2$, $[u, v] = \varepsilon(f)$ where

$f(x) = \begin{cases} u & \text{if } x = s \\ v & \text{if } x = t \\ 0 & \text{otherwise} \end{cases}$

$\therefore$ $\dfrac{R}{J} \cong F \times F$

Since $F \times F$ is not an integral domain ($[1,0] \cdot [0,1] = [0,0]$)

J is not prime ☺

3. Find the quotient and remainder for $x^5 + 4x^3 + 2x^2 + 3$ divided by $x + 6$ in $\mathbf{Z}_7[x]$.

$$\boxed{x^4 + x^3 + 5x^2}$$ quotient

$$x+6 \,\big)\, x^5 + 4x^3 + 2x^2 + 3$$

$$x^5 + 6x^4$$

$$\underbrace{-6}_{1}x^4 + 4x^3 + 2x^2 + 3$$

$$x^4 + 6x^3$$

$$\underbrace{-2}_{5}x^3 + 2x^2 + 3$$

$$5x^3 + \underbrace{30}_{2}x^2$$

$$3$$ remainder

4. Suppose F is a field and $s \in F$. Let $I = \{f \in F[x] \colon f(s) = 0\}$. Use the division algorithm to prove that I is the ideal generated by $x - s$.

Since $x - s \Big|_{x=s} = 0$, $x - s \in I$

Let $f \in I$. Div. Alg. $\Rightarrow \exists!\, q, r \in F[x]$ s.t.

$f(x) = q(x)(x-s) + r(x)$, $r(x) = 0$ or $\deg r < \deg(x-s)$

If $r(x) = 0$, $f(x) = q(x)(x-s) \in \langle x-s \rangle$ done

if $\deg r < \deg(x-s) = 1$, $\deg r = 0$, so $r = \text{const.}$

Substitute $x = s$ into $f(x) = q(x)(x-s) + r$

Since $f(s) = 0$, $r = 0$ so done ☺

5. Let J be the ideal generated by x and 2 in $\mathbf{Z}[x]$. Prove that J is a maximal ideal.

Suppose K is an ideal of $\mathbb{Z}[x]$ s.t. $J \subsetneq K$

Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in K \setminus J$

If a_0 is even, $\exists k \in \mathbb{Z}$ $a_0 = 2k$, so

$f(x) = \underline{2}k + \underline{x}(a_1 + a_2 x + \cdots) \in J$ ☹

$\therefore$ a_0 is odd, so $\exists k \in \mathbb{Z}$ $a_0 = 2k+1$. Then

$\underbrace{f(x)}_{\in K} = 1 + \underbrace{2k + x(a_1 + \cdots)}_{\in J \subset K}$ $\therefore 1 \in K$

$\therefore K = \mathbb{Z}[x]$

$\therefore$ J is maximal ☺

<u>With homs</u> Let $\varepsilon : \mathbb{Z}[x] \to \mathbb{Z}$ be the evaluation hom $\varepsilon(f) = f(0)$

Let π be the usual projection $\mathbb{Z} \to \mathbb{Z}_2$

Compose: $\mathbb{Z}[x] \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2$

Clearly onto and $\ker(\pi\varepsilon) = J$

By 1^{st} iso. thm $\dfrac{\mathbb{Z}[x]}{J} \cong \mathbb{Z}_2$

Since $\mathbb{Z}_2$ is a field, J is a max. ideal of $\mathbb{Z}[x]$ ☺