1. Let $G = \mathrm{Gl}(n, \mathbf{R})$ be the multiplicative group of all invertible $n \times n$ real matrices, K a multiplicative subgroup of nonzero real numbers $\mathbf{R}^*$ and $H = \{A \in G\colon \det A \in K\}$. Prove that H is a normal subgroup of G.

$$\det : Gl(n, \mathbb{R}) \longrightarrow \mathbb{R}^*$$

(if $A \in Gl(n, \mathbb{R})$, then $\det A \neq 0$

is a group hom: $\det(AB) = \det A \cdot \det B$

$H = \{ A \in Gl(n, \mathbb{R}), \det A \in K \}$

= inverse image of K under "det" map.

Since $K < \mathbb{R}^*$ is normal ($\mathbb{R}^*$ is comm.), $H \lhd G$.

* $\det(\text{Id matrix}) = 1 \in K$ (since $K < \mathbb{R}^*$) $\therefore \mathrm{Id} \in H$
* Let $A, B \in H$. Then $\det A \in K$, $\det B \in K$

since $K < \mathbb{R}^*$, $\underbrace{(\det B)^{-1}}_{\det(B^{-1})} \in K$

$\det(AB^{-1}) = \underbrace{\det A}_{\in K} \cdot \underbrace{\det B^{-1}}_{\in K} \in K$

$\therefore H < Gl(n, \mathbb{R})$

Suppose $A \in Gl(n, \mathbb{R})$, $B \in H$

$\det(ABA^{-1}) = \cancel{\det A} \cdot \det B \cdot \cancel{(\det A)^{-1}} = \det B \in K$

$\therefore H \lhd Gl(n, \mathbb{R})$

Alt: $Gl(n, \mathbb{R}) \xrightarrow{\det} \mathbb{R}^* \xrightarrow{\pi} \mathbb{R}^*/K$

$\ker \pi = K$, so $\ker(\det \circ \pi) = H$:)

2. Find (with proof) a group homomorphism on the symmetric group S_n of all permutations on n elements (to a suitable range), whose kernel is the alternating subgroup A_n of all even permutations.

Define $\phi : S_n \to \mathbb{Z}_2$ by

$$\phi(\tau) = \begin{cases} 0 & \text{if } \tau \in A_n \\ 1 & \text{otherwise} \end{cases}$$

Claim: ϕ is a group hom

Let $\delta, \tau \in S_n$. Consider cases:

1. δ, τ both even
2. δ even, τ odd
3. δ odd, τ even
4. δ odd, τ odd

Case 1. Since δ, τ are even, we can write

$\delta = \alpha_1 \alpha_2 \dots \alpha_n$, $\tau = \beta_1, \beta_2 \dots \beta_m$

where n & m are even

Then $\delta\tau = \underbrace{\alpha_1, \alpha_2 \dots \alpha_n \beta_1 \dots \beta_m}_{n+m}$ is even

(even + even = even)

$\phi(\delta\tau) = 0$ $\quad\quad$ $\phi(\delta) + \phi(\tau) = 0 + 0 = 0$ ✓

Case 2 With the same notation assume m is odd.

even + odd = odd

$\phi(\delta\tau) = 1$ $\quad\quad$ $\phi(\delta) + \phi(\tau) = 0 + 1 = 1$ ✓

Case 3 similar to Case 2

Case 4 $\quad$ odd + odd = even

$\phi(\delta\tau) = 0$ $\quad\quad$ $\phi(\delta) + \phi(\tau) = 1 + 1 = 0$ ✓ ◡̈

3. Let R be the ring of all functions $\mathbf{R} \to \mathbf{R}$ with pointwise operations and let $a \in \mathbf{R}$. Prove that $I = \{f \in R\colon f(a) = 0\}$ is a maximal ideal of R.

Pf 1 * if $f \equiv 0$, then $f(a) = 0$
so the zero function $\in I$

* Suppose $f, g \in I$, then $f(a) = g(a) = 0$,
so $(f-g)(a) = f(a) - g(a) = 0 - 0 = 0$
If $f \in I, g \in R$ $(fg)(a) = f(a)g(a) = 0 \cdot g(a) = 0$
$\therefore$ I is an ideal of R.

Suppose J is an ideal of R, $I \subsetneq J$.
Let $f \in J \setminus I$. Since $f \notin I$, $f(a) \neq 0$.
Define g by $g(x) = f(a)$ (const. function)

Then $g(x) = \underbrace{f(a) - f(x)}_{\substack{\text{Plug in } a \\ \text{and you have } 0, \\ \text{so } \in I \subset J}} + \underbrace{f(x)}_{\in J} \in J$

Since $g(x)$ is a unit in R, $J = R$. $\therefore$ I is maximal.

Pf2 Define $\varepsilon : R \to \mathbb{R}$ by $\varepsilon(f) = f(a)$
ε is a hom: $\varepsilon(fg) = (fg)(a) = f(a) \cdot g(a) = \varepsilon(f) \cdot \varepsilon(g)$
$\ker \varepsilon = I$ By 1^{st} iso. thm $\dfrac{R}{I} \cong im(\varepsilon) = \underset{\substack{\uparrow \\ \text{field}}}{\mathbb{R}}$

$\therefore$ I is a max ideal of R.

4. With R as in the preceding problem, let $a \neq b \in \mathbf{R}$. Prove that $J = \{f \in R\colon f(a) = 0, f(b) = 0\}$ is an ideal of R that is not prime. Find (with proof) two prime ideals containing J.

4 Pf 1 * If $f \equiv 0$, $f(a) = 0$ and $f(b) = 0$
so the zero function $\in J$.

* Given $f, g \in J$, $(f-g)(a) = f(a) - g(a) = 0 - 0 = 0$
and similarly for b, so $f - g \in J$

Given $f \in J$, $g \in R$, $(fg)(a) = f(a)g(a) = 0 \cdot g(a) = 0$
similarly for b, so $fg \in J$.

$\therefore$ J is an ideal of R.

$x - a \notin J$, $x - b \notin J$, but $(x-a)(x-b) \in J$

$\therefore$ J is not prime :)

$J \subseteq \{f \in R : f(a) = 0\}$ which is a maximal ideal by #3
similarly for b. :)

Pf 2 Define $\varepsilon : R \longrightarrow \mathbb{R}^2$ by $\varepsilon(f) = [f(a), f(b)]$

* ε is a hom (easy to show, similar to #3) :)

* ε is onto: Given $[r, s] \in \mathbb{R}^2$, let $f(x) = \frac{s-r}{b-a}(x-a) + r$.
Then $f(a) = r$, $f(b) = s$, so $\varepsilon(f) = [r, s]$:)

* $J = \ker \varepsilon$

* By the 1st isomorphism theorem, $\frac{R}{J} \cong \mathbb{R}^2$ not an integral domain ($[1,0] \cdot [0,1] = [0,0]$)

$\therefore$ J is not prime.

5. Suppose F is a field and p is polynomial in $F[x]$ of degree 3. Prove that p irreducible if and only if p has no roots in F.

Suppose $p(a) = 0$ for some $a \in F$

Long div: $\exists !\ q(x), r(x)$ s.t.

$$p(x) = (x-a)q(x) + r(x)$$

$r = 0$ or $\deg r < \deg (x-a) = 1$

(done)

plug in a

$\deg r = 0$

$r = \text{const}$

$0 = p(a) = 0 \cdot q(a) + r$

$\therefore r = 0$

$\therefore p(x) = (x-a)q(x)$

$\therefore p$ is reducible.

Conversely suppose p is reducible, then

$\exists\ r, s \in F[x]$ $p = r \cdot s$ $\deg r, \deg s \neq 0$

Possibilities for degrees $(\deg r + \deg s = 3)$

r	s
2	1
1	2

$\therefore \deg r = 1$ or $\deg s = 1$

If $\deg r = 1$, r has a zero, so p has a zero

Similarly if $\deg s = 1$. ☺