(1) $\tau = (1,4,5)(1,3,5,2)$

$$= \underset{\uparrow}{(1,3)(2,4,5)}$$

disjoint cycles commute, so

$$\tau^{2015} = (1,3)^{2015}(2,4,5)^{2015}$$

$$= (1,3)^{2014+1}(2,4,5)^{2016-1} =$$

$$= \underbrace{(1,3)^{2014}}_{\varepsilon}(1,3)\underbrace{(2,4,5)^{2016}}_{\varepsilon}(2,5,4)$$

$$= (1,3)(2,5,4)$$

(2) $A_3 \triangleleft \Sigma_3$

<u>Proof</u> ε is even, If σ, τ are even, so is $\sigma\tau^{-1}$

$$\underbrace{\underset{\substack{2k \\ \text{2-cycles}}}{\sigma}\ \underset{\substack{2\ell \\ \text{2-cycles}}}{\tau^{-1}}\quad \tau^{-1} \text{ is still } 2\ell \text{ 2-cycle}}_{2(k+\ell)\ -\ \text{even} ☺}$$

$A_3 < \Sigma_3$

If $\sigma \in A_3$, $\tau \in \Sigma_3$, then

$$\tau\sigma\tau^{-1} \in A_3$$

$$\underbrace{\underset{m}{\tau}\ \underset{2k}{\sigma}\ \underset{m}{\tau^{-1}}}_{2(k+m)\ -\ \text{even} ☺}$$

$\langle (1,2) \rangle < \Sigma_3$, but $\langle (1,2) \rangle \not\trianglelefteq \Sigma_3$

$\langle (1,2) \rangle = \{\varepsilon, (1,2)\}$

$$(2,3)(1,2)(2,3)^{-1} = (2,3)(1,2)(2,3)$$
$$= (1,3) \notin \langle (1,2) \rangle \quad ☺$$

(3) $\phi(3) = [1,5] \in \mathbb{Z}_2 \oplus \mathbb{Z}_7$

$\phi(3) = 3\phi(1)$

$3 \cdot 5 = 15 \equiv 1 \bmod 14$

$\therefore \ \phi(1) = 5[1,5] = [5,25] \equiv [1,4]$

(4) Let $I = \{f : X \to \mathbb{R} \, ; \, f(0) = 0\}$

1. $0 \in I$ ☺
2. If $f, g \in I$ then $f(0) = g(0) = 0$
   so $(f-g)(0) = f(0) - g(0) = 0$
   $\therefore \ f - g \in I$
3. If $f \in I$, $g \in R$, then
   $(fg)(0) = f(0)g(0) = 0 \cdot g(0) = 0$
   $\therefore \ fg \in I$

$\therefore$ I is an ideal of R.

Suppose J is an ideal of R s.t. $I \subsetneq J$

Let $g \in J \setminus I$. Then $g(0) \neq 0$

Let f be given by

$$\begin{array}{c|cc} x: & 0 & 1 \\ f(x): & 0 & 1-g(1) \end{array}$$

Then $f \in I \subset J$, so $h = g + f \in J$

Now h looks like

$$\begin{array}{c|cc} x: & 0 & 1 \\ h(x): & \underset{\neq 0}{g(0)} & \underset{\neq 0}{1} \end{array}$$

So h is a unit, so $J = R$.

$\therefore$ I is a maximal ideal of R.

<u>Alt. proof</u>: Define $\phi: R \to \mathbb{R}$ by

$$\phi(f) = f(0)$$

ϕ is a ring hom:

$$\phi(f+g) = (f+g)(0) = f(0) + g(0) = \phi(f) + \phi(g)$$

$$\phi(f \cdot g) = (f \cdot g)(0) = f(0) \cdot g(0) = \phi(f) \cdot \phi(g)$$

$\ker \phi = \{f : \phi(f) = 0\} = \{f : f(0) = 0\} = I$

ϕ is onto: If $r \in \mathbb{R}$ $\phi(r) = r$

$\uparrow$ const. function r

$$\begin{array}{c|cc} x: & 0 & 1 \\ & r & r \end{array}$$

By the 1^{st} isomorphism theorem:

$$R/I \cong \mathbb{R} \quad \text{field}$$

$\therefore$ I is a maximal ideal. ☺

(5) Suppose x, y are associates, i.e. $x = yu$, where u is a unit.

Then $x \in \langle y \rangle$, so $\langle x \rangle \subseteq \langle y \rangle$

Since $y = xu^{-1}$, similarly $\langle y \rangle \subseteq \langle x \rangle$

$\therefore$ $\langle x \rangle = \langle y \rangle$ ☺

Conversely, suppose $\langle x \rangle = \langle y \rangle$.

If $x = 0$, then $\langle y \rangle = \{0\}$, so $y = 0$, so might as well assume $x \neq 0$ and $y \neq 0$.

Since $\langle x \rangle = \langle y \rangle$, $x \in \langle y \rangle$, $y \in \langle x \rangle$, so

$\exists\, a, b$ $\quad x = ya \quad y = xb$,

so $\cancel{x} = \cancel{x} ba$ (R is a domain)

so $ba = 1$ so a & b are units,

so a and b are associates. ☺