(1) $\langle m \rangle \cap \langle n \rangle = \langle \operatorname{lcm}(m,n) \rangle$

Let $k \in \langle m \rangle \cap \langle n \rangle$

then $k \in \langle m \rangle = \{ im : i \in \mathbb{Z} \} = m\mathbb{Z}$

so $k$ is a multiple of $m$

and similarly of $n$, so

$\operatorname{lcm}(m,n) \mid k$, so $k \in \langle \operatorname{lcm}(m,n) \rangle$

Conversely if $k \in \langle \operatorname{lcm}(m,n) \rangle$

then $\operatorname{lcm}(m,n) \mid k$

so both $m$ & $n$ divide $k$

so $k \in \langle m \rangle$ and $k \in \langle n \rangle$

so $k \in \langle m \rangle \cap \langle n \rangle$.

(2) $\tau = (2\,4\,5)(1\,3\,5\,2)$ $\qquad \tau^{11} = ?$

$= (1\,3\,2)(4\,5)$

Since disjoint cycles commute,

$\tau^{11} = (1\,3\,2)^{11}\,(4\,5)^{11}$

$= (1\,2\,3)(4\,5)$

$(1\,3\,2)^{11} = (\underbrace{(1\,3\,2)^3}_{()})^4 (1\,3\,2)^{-1}$

(3) $\phi : \mathbb{Z}_m \to \mathbb{Z}_m \qquad \phi(x) = ax$

a) Hom: $\phi(x+y) = a(x+y) = ax + ay$
$= \phi(x) + \phi(y)$ ☺

If $a$ is a unit, then $\phi$ is invertible
$(\phi^{-1}(x) = a^{-1}x)$

If $\phi$ is an automorphism, it's onto,
so $\exists x \quad \phi(x) = 1$, i.e. $ax = 1$
$\therefore a$ is unit. ☺

b) $\phi(0) = 0$ ☺ $(a0 = 0) \qquad (a = \phi(1))$
$\phi(1) = \phi(1) \qquad (a \cdot 1 = a)$
$\phi(2) = \phi(1+1) = \phi(1) + \phi(1) = 2\phi(1)$
etc.
$\phi(-1) + \phi(1) = \phi(-1+1) = \phi(0) = 0$
so $\phi(-1) = -\phi(1)$
$\phi(-2) = \phi(-1-1) = \phi(-1) + \phi(-1)$
$= -\phi(1) - \phi(1) = -2\phi(1)$ ☺
etc.

Shortcut: $\phi(k) = k\phi(1)$

c) $\theta : \text{Aut}\, \mathbb{Z}_m \longrightarrow U(m)$

Given $\phi \in \text{Aut}\, \mathbb{Z}_m$, by (a-b)

$\phi(x) = ax$ for some unit $a$

so define $\theta$ by $\theta(\phi) = a = \phi(1)$

Hom: $\theta(\phi \circ \psi) = (\phi \circ \psi)(1) = \phi(\psi(1))$

$= \psi(1)\phi(1)$ ☺

Onto: Given $a \in U(m)$, define

$\phi$ by $\phi(x) = ax$, then $\theta(\phi) = a$ ☺

1-1: Suppose $\theta(\phi) = 1$,

then $\phi(1) = 1$, so $\phi(x) = x\phi(1) = x \cdot 1 = x$

$\therefore$ ker $\theta$ is trivial ☺

4. $\langle 11 \rangle = \{1, 11\}$

$U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$

$3\langle 11 \rangle = \{3, 13\}$

$7\langle 11 \rangle = \{7, 17\}$

$9\langle 11 \rangle = \{9, 19\}$