Modern Abstract Algebra MAT 4233 Midterm 2, Sp '13

(1) $H = \{[0,0], [2,0], [0,2], [2,2]\}$

$K = \{[0,0], [1,2], [2,0], [3,2]\}$

By Lagrange's theorem each subgp. has 4 cosets

Cosets: $[1,0] + H = \{[1,0], [3,0], [1,2], [3,2]\}$

$[0,1] + H = \{[0,1], [2,1], [0,3], [2,3]\}$

$[1,1] + H = \{[1,1], [3,1], [1,3], [3,3]\}$

$[1,0] + K = \{[1,0], [2,2], [3,0], [0,2]\}$

$[0,1] + K = \{[0,1], [1,3], [2,1], [3,3]\}$

$[1,1] + K = \{[1,1], [2,3], [3,1], [0,3]\}$

By the classification of finite abelian groups

$\frac{G}{H}$ and $\frac{G}{K} \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ which?

Compute orders: $[1,0] + H \to 2$

$[0,1] + H \to 2$

$[1,1] + H \to 2$ $\quad \therefore \frac{G}{H} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$[0,1] + K \to 4 \quad \therefore \frac{G}{K} \cong \mathbb{Z}_4$ ☺

(2) $e \in H, e \in K \quad \therefore \quad e = e \cdot e \in HK$

Let $h, h' \in H$, $k, k' \in K$.

Then $hkh'k' = h\underbrace{\underbrace{k h' k^{-1}}_{\in H \text{ (since } H \trianglelefteq G)}}_{\in H}\underbrace{k k'}_{\in K} \in HK$

$\therefore HK < G$ (since G is finite, inverses are automatic)

Let $x \in G$. Then

$x^{-1}HKx = \underbrace{x^{-1}Hx}_{\in H}\underbrace{x^{-1}Kx}_{\in K} \in HK \quad \therefore HK \trianglelefteq G$ ☺

(3) Let $\phi : \mathbb{Z}_{11} \to G$ be a group hom.

Then $\ker \phi < \mathbb{Z}_{11}$. By Lagrange's theorem

$|\ker \phi|$ divides 11, so $|\ker \phi| = 1$ or 11

Since ϕ is not injective, $\ker \phi \neq \langle 0 \rangle$, so

$|\ker \phi| = 11$, so $\ker \phi = \mathbb{Z}_{11}$

$\therefore \forall x \in \mathbb{Z}_{11} \quad \phi(x) = e$ ☺

(4) $ba = (ba)^n = \underbrace{b\overbrace{ab}^{=0}a \dots ba}_{n} = 0$

(5) Suppose $I \subsetneq J \leftarrow$ an ideal of R

Ideals are additive subgroups, so by Lagrange's theorem

$\exists m, n \in \mathbb{N}$ $|R| = m|J|$ and $|J| = n|I|$, so $|R| = mn|I|$, i.e. $500 = mn100$,

so $mn = 5$. Since 5 is prime, $m = 1$ and $n = 5$ or vice versa.

$\therefore |J| = 100$ or 500, so $J = I$ or R

$\therefore I$ is a max. ideal of R, so $\frac{R}{I}$ is a field. ☺