Midterm 2 / 2010.4.26 / MAT 4233.001 / Modern abstract algebra

Note Title 4/26/2010

1. Find the isomorphism class of $U(12)$ as a finite abelian group.

$U(12) = \{x \in \mathbb{Z}_{12} : \gcd(x, 12) = 1\} = \{1, 5, 7, 11\}$

By the classification theorem $U(12) \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Since $5^2 \equiv 7^2 \equiv 11^2 \equiv 1 \bmod 12$, $U(12)$ has no elements of order 4, so $U(12) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

2. Find all ideals of $\mathbf{Z}_{60}$. Explain why that's all of them. Draw a lattice (i.e. sketch subset relations among the ideals).

Since $\mathbb{Z}_{60}$ is a cyclic additive group, its ideals are cyclic subgroups and correspond to the divisors of 60

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> numtheory[divisors](60);
                    {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}
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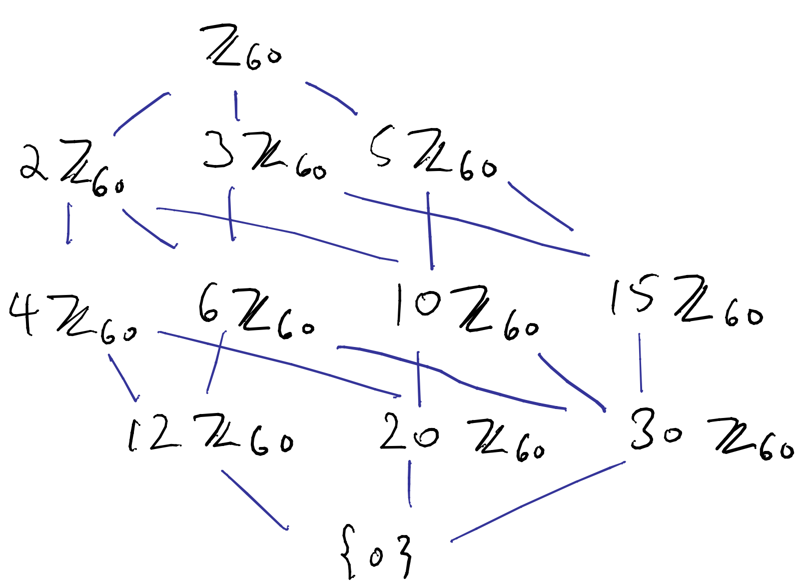

$H=$

3. Prove that $\{\sigma \in S_3 \colon \sigma(3) = 3\}$ is a subgroup of S_3. Is it abelian? Is it a normal subgroup of S_3? Prove your assertions.

a) $()$ takes 3 to 3, so $() \in H$

b) if $\sigma, \tau \in H$, then $\sigma(3)=3$, $\tau(3)=3$,
so $\sigma\tau(3) = \sigma(\tau(3)) = \sigma(3) = 3$ so $\sigma\tau \in H$

c) If $\sigma \in H$, then $\sigma(3)=3$, so $\sigma^{-1}(3)=3$, so $\sigma^{-1} \in H$.

$H = \{(), (12)\}$ is abelian $(H \cong \mathbb{Z}_2)$

Let $\tau = (23)$. Then $\tau^{-1} = (23)$

$(23)(12)(23) = (13) \notin H$, so $\tau^{-1}H\tau \not\subseteq H$, so $H \ntrianglelefteq S_3$

4. Find the quotient and remainder of $x^4 + 3x^3 + 2x^2 + x - 1$ divided by $2x^2 + 1$ in $\mathbf{Z}_7[x]$.

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> p:=x^4+3*x^3+2*x^2+x-1;
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$$p := x^4 + 3x^3 + 2x^2 + x - 1$$

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> q:=2*x^2+1;
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$$q := 2x^2 + 1$$

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> quo(p,q,x,'r') mod 7; r mod 7;
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$$4x^2 + 5x + 6$$

$$3x$$

5. Let $A = \{p \in \mathbf{R}[x] \colon p(0) = 0\}$. Prove that A is an ideal of $\mathbf{R}[x]$. Is A is a prime ideal? Maximal? Explain.

Define $\varphi : \mathbb{R}[x] \to \mathbb{R}$ by $\varphi(p(x)) = p(0)$

By inspection φ is a ring homomorphism with $\ker \varphi = A$ and $\varphi(\mathbb{R}[x]) = \mathbb{R}$

By the 1st isomorphism theorem $\dfrac{\mathbb{R}[x]}{A} \cong \mathbb{R}$

Since $\mathbb{R}$ is a field, A is a maximal ideal (thus a prime ideal) of $\mathbb{R}[x]$.