Midterm 2 / 2009.4.23 / MAT 4233.001 / Modern abstract algebra

Note Title 4/23/2009

1. Prove that $A_n \triangleleft S_n$ and that $S_n/A_n \cong \mathbb{Z}_2$.

① $() \in A_n$ (because 0 is even :)

② Since the inverse of a product of transpositions, can be expressed as the product of the same transpositions in reverse order, $\tau \in A_n \Rightarrow \tau^{-1} \in A_n$. ③ Also $\sigma, \tau \in A_n \Rightarrow \sigma\tau \in A_n$ $\left(\substack{\text{even}+\text{even} \\ =\text{even}}\right)$, so $A_n < S_n$

④ If $\sigma \in A_n$ and $\tau \in S_n$, then $\tau\sigma\tau^{-1}$ has an even # of transpositions: those for σ (even) and twice those for τ, so $A_n \triangleleft S_n$

Alt Since the index of A_n in S_n is 2, there is only one nontrivial coset, so left coset = right coset, so $A_n \triangleleft S_n$

$\left|\frac{S_n}{A_n}\right| = \text{index} = 2$, But all groups of order 2 are $\cong \mathbb{Z}_2$.

Alt. Define $\varphi: S_n \to \mathbb{Z}_2$ By $\varphi(\tau) = \begin{cases} 0 & \text{if } \tau \text{ is even} \\ 1 & \text{if } \tau \text{ is odd} \end{cases}$

Then φ is an onto hom and $\ker \varphi = A_n$

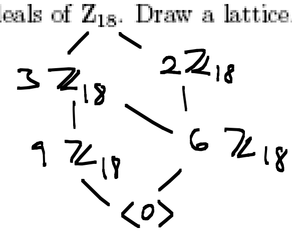
Now apply the First Isomorphism Theorem.

2. Let A be the multiplicative group $\{1, 9, 16, 22, 29, 53, 74, 79, 81\} \subset \mathbb{Z}_{91}$. Find the isomorphism class of A as a finite abelian group.

Since $|A| = 9$, $A \cong \mathbb{Z}_9$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3$

Since A has no elements of order 9, $A \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$

3. Find all ideals of $\mathbb{Z}_{18}$. Draw a lattice.



As an additive subgroup $\mathbb{Z}_{18}$ is cyclic So its subgroups correspond to divisors of 18.

4. Let $A = \{p \in \mathbb{Z}[x]: p(0) = 0\}$. Prove that A is an ideal of $\mathbb{Z}[x]$. Prove that A is a prime ideal, but not maximal.

$(A = \langle x \rangle)$

① $0(0) = 0$, So $0 \in A$

② If $p, q \in A$, then $p(0) = 0$ and $q(0) = 0$, So $(p-q)(0) = p(0) - q(0) = 0 - 0 = 0$ so $p - q \in A$

③ If $p \in A$, $q \in \mathbb{Z}[x]$, then $p(0) = 0$, so $(pq)(0) = p(0)q(0) = 0 \cdot q(0) = 0$ so $pq \in A$

$\therefore$ A is an ideal

④ If $pq \in A$, $(pq)(0) = p(0)q(0) = 0$, then $p(0) = 0$ or $q(0) = 0$ so $p \in A$ or $q \in A$ $\therefore$ A is a prime ideal.

(5) Let $I = \{p \in \mathbb{Z}[x] : p(0) \text{ is even}\}$, then $A \subsetneq I \subsetneq \mathbb{Z}[x]$

$(I = \langle 2, x \rangle)$

$\therefore$ A is not maximal

Alt. Define $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ By $\varphi(p) = p(0)$

Then φ is an onto ring hom with $\ker\varphi = A$

By the First Isomorphism Theorem

$\frac{\mathbb{Z}[x]}{A} \cong \mathbb{Z}$ which is an integral domain

(so A is a prime ideal)

But not a field

(so A is not maximal)

5. Let $A = \{(x, y) \in \mathbf{Z} \oplus \mathbf{Z}: y \text{ is even}\}$. Prove that A is an ideal of $\mathbf{Z} \oplus \mathbf{Z}$. Prove that A is maximal.

(1) Since 0 is even, $(0,0) \in A$

(2) If $(x,y), (x',y') \in A$, then y and y' are even, so $y-y'$ is even,

so $(x,y) - (x',y') = (x-x', y-y') \in A$

(3) If $(x,y) \in A$, $(x',y') \in \mathbb{Z} \oplus \mathbb{Z}$, then y is even, so yy' is even,

so $(x,y)(x',y') = (xx', yy') \in A$

$\therefore$ A is an ideal.

(4) Suppose I is an ideal of $\mathbb{Z} \oplus \mathbb{Z}$ such that $A \subsetneq I$

Let $(x,y) \in I \setminus A$, then y is odd, so $y-1$ is even,

So $(x-1, y-1) \in A \subset I$,

so $\underset{\text{unit}}{(1,1)} = \underbrace{(x,y)}_{\in I} - \underbrace{(x-1, y-1)}_{\in I} \in I$

so $I = \mathbb{Z} \oplus \mathbb{Z}$

$\therefore$ A is maximal

Alt Define $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2$ By

$$\varphi(x,y) = \begin{cases} 0 & \text{if } y \text{ is even} \\ 1 & \text{if } y \text{ is odd} \end{cases}$$

Then φ is an onto ring hom with $\ker\varphi = A$

By F.I.T. $\frac{\mathbb{Z} \oplus \mathbb{Z}}{A} \cong \mathbb{Z}_2$ which is a field, so A is maximal.