Final / 2009.5.8 / MAT 4233.001 / Modern abstract algebra

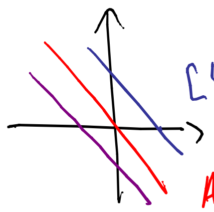
Note Title 4/23/2010

1. Let $A = \{(x,y) \in \mathbf{R}^2 \colon x + y = 0\}$. Prove that A is an additive subgroup of $\mathbf{R}^2$. Sketch A and two different nontrivial cosets of A in $\mathbf{R}^2$.

Since $0+0=0$ $[0,0] \in A$

if $[x,y], [x',y'] \in A$, then $x+y = x'+y' = 0$

So $(x-x')+(y-y') = 0$ so $[x,y]-[x',y'] \in A$:)



2. Suppose G is a group with 81 elements. Prove that G has an element of order 3. Provide an explicit example to show that G need not have an element of order 9.

Let $x \in G$, $x \neq e$, then $|x| \neq 1$ and divides $81 = 3^4$

by Lagrange's theorem, so $|x| = 3^k$ for some $1 \leq k \leq 4$

We have $e = x^{3^k} = (x^{3^{k-1}})^3$, so $|x^{3^{k-1}}|$ divides 3

Since $x^{3^{k-1}} \neq e$ by the minimality of k, $|x^{3^{k-1}}| = 3$

Example $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ has order 81

But no element of order 9.

3. Let $Gl_2(\mathbf{R})$ denote the multiplicative group of invertible 2×2 matrices with real coefficients and $Sl_2(\mathbf{R})$ denote the subgroup of $Gl_2(\mathbf{R})$ of those matrices with determinant 1. Prove that $Sl_2(\mathbf{R}) \triangleleft Gl_2(\mathbf{R})$ and $Gl_2(\mathbf{R})/Sl_2(\mathbf{R}) \cong \mathbf{R}^*$.

Define $\varphi : Gl_2(\mathbb{R}) \to \mathbb{R}$

By $\varphi(A) = \det A$

Then since $\det(AB) = \det A \cdot \det B$, φ is group hom.

$\ker \varphi = Sl_2(\mathbb{R})$ so $Sl_2(\mathbb{R}) \triangleleft Gl_2(\mathbb{R})$ (kernels are always normal subgroups)

1st iso. thm: $\dfrac{Gl_2(\mathbb{R})}{Sl_2(\mathbb{R})} \cong \varphi(Gl_2(\mathbb{R})) = \mathbb{R}^*$

4. Find the isomorphism class of $U(5)$ as a finite abelian group. Explain your reasoning.

$U(5) = \{1, 2, 3, 4\}$ so $U(5) \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

$2^2 = 4 \therefore |2| \neq 2 \therefore U(5) \cong \mathbb{Z}_4$

(so must be 4)

5. Prove that a finite integral domain must be a field.

Let R be a finite integral domain and let $x \in R$, $x \neq 0$.

Since R is finite not all powers of x are distinct.

So $\exists n > m$ $x^n = x^m$, i.e. $x^n - x^m = 0$

$x^m(x^{n-m} - 1) = 0$

Since R is an integral domain and $x \neq 0$, $\underbrace{x^{n-m}}_{>0} = 1$

so $x \cdot x^{n-m-1} = 1$, so x is a unit ☺

$(x^{-1} = x^{n-m-1})$

6. Prove or disprove that $\mathbf{Z}_2[x]$ has infinitely many ideals.

$\langle x \rangle, \langle x^2 \rangle, \langle x^3 \rangle, \ldots$ are distinct ideals.

7. Let $A = \{p \in \mathbf{Z}_m[x] \colon p(0) = 0\}$. Prove that A is an ideal of $\mathbf{Z}_m[x]$ and $\mathbf{Z}_m[x]/A \cong \mathbf{Z}_m$.

Define $\varphi : \mathbb{Z}_m[x] \to \mathbb{Z}_m$ by $\varphi(p(x)) = p(0)$

By inspection φ is an onto ring homomorphism with $\ker \varphi = A$. By the 1st isomorphism theorem

$$\frac{\mathbb{Z}_m[x]}{A} \cong \varphi(\mathbb{Z}_m[x]) = \mathbb{Z}_m$$

8. Let A be as in the preceding problem. Prove that A is a principal ideal, i.e. can be generated by just one polynomial. For which m is A a prime ideal of $\mathbf{Z}_m[x]$. For which m is A maximal? Explain.

Since $\mathbb{Z}_m$ is a field for m prime and not an integral domain for composite m (if $m = rs$, then $r \cdot s \equiv 0 \bmod m$)

A is prime $\Leftrightarrow$ A is maximal $\Leftrightarrow$ m is prime

$\Updownarrow$ $\mathbb{Z}_m$ is an integral domain $\quad$ $\Updownarrow$ $\mathbb{Z}_m$ is a field