Midterm 1 / 2008.3.6 / Modern abstract algebra / MAT 4233.001

Note Title 3/6/2008

1. Suppose G is a group such that every nontrivial element of G has order 2. Prove that G is abelian. Give an example of such a group that is not isomorphic to $\mathbf{Z}_2$.

$\forall x \in G \quad x^2 = e$, so $x = x^{-1}$.

Let $a, b \in G$, then $\underline{ab} = (ab)^{-1} = b^{-1}a^{-1} = \underline{ba}$

Example: $\underline{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$

2. Prove that a group whose order is a prime must be cyclic.

Suppose $|G| = p$ (a prime). Since $p > 1$, G has a nontrivial element x.

By Lagrange's theorem $|x| = |\langle x \rangle|$ divides $|G| = p$.

Since $x \neq e$, $|x| \neq 1$ so $|x| = p$.

$\therefore \underline{\langle x \rangle = G}$

3. Let $H = \{\alpha \in S_n : \alpha(1) = 1\}$ with $n \geq 5$. Prove that H is a subgroup of S_n. Prove or disprove that H a normal subgroup of S_5.

Identity: $\varepsilon(1) = 1$

Closure: If $\alpha, \beta \in H$, then $\alpha(1) = \beta(1) = 1$, so
$\alpha\beta(1) = \alpha(\beta(1)) = \alpha(1) = 1$. $\therefore \alpha\beta \in H$

Inverses: If $\alpha \in H$, then $\alpha(1) = 1$, so $\alpha^{-1}(1) = 1$, so $\alpha^{-1} \in H$

$\therefore$ <u>H is a subgroup of S_n</u>

Extra credit: <u>H is not normal in S_n.</u>

Let $\alpha = (23)$. Then $\alpha(1) = 1$ so $\underline{\alpha \in H}$

Let $\gamma = (12)$. Then $\gamma\alpha\gamma^{-1}(1) = \gamma(\alpha(\gamma^{-1}(1)))$
$= \gamma(\alpha(2)) = \gamma(3) = 3$, so $\underline{\gamma\alpha\gamma^{-1} \notin H}$

4. Let H be as in the preceding problem. Suppose $\beta, \gamma \in S_n$ with $\beta(1) = \gamma(1)$. Prove that β and γ belong to the same left coset of H.

$\beta = \gamma \underline{\gamma^{-1}\beta}$, But $\gamma^{-1}\beta(1) = \gamma^{-1}(\beta(1)) = \gamma^{-1}(\gamma(1)) = 1$

So $\gamma^{-1}\beta \in H$. $\therefore$ $\beta \in \gamma H$ ☺

5. Suppose G is an abelian group whose order is odd. Prove that $\varphi : G \to G$ given by $\varphi(x) = x^2$ is an automorphism of G.

$\varphi(xy) = (xy)^2 = xyxy \underset{\uparrow \text{abelian}}{=} xxyy = x^2y^2 = \varphi(x)\varphi(y)$

$\therefore \varphi$ is a <u>hom</u>.

Suppose $\varphi(x) = x^2 = e$. By Lagrange's theorem

$|x| = |\langle x \rangle|$ divides $|G|$, but $|G|$ is odd,

so $|x| \neq 2$, so $|x| = 1$, so $x = e$.

$\therefore \varphi$ is <u>1-1</u>

Since G is finite, φ is automatically <u>onto</u>. ☺

$\therefore \varphi$ <u>$\in \mathrm{Aut}\, G$</u>