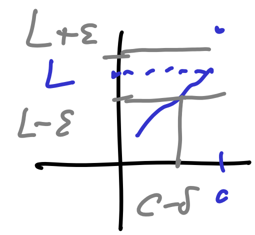
① $f: \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$, f increasing

Prove $\lim\limits_{x \to c^-} f(x)$ exists.



Let $S = \{ f(x) : x < c \}$

$c-1 < c$, so $f(c-1) \in S$, so $S \neq \phi$

Since f is incr. $x < c \Rightarrow f(x) < f(c)$

so S is bdd. above by $f(c)$.

$\mathbb{R}$ is complete so $\exists \sup S$, let $L = \sup S$

Given $\varepsilon > 0$, $L - \varepsilon$ is not an upper bound for S.

so $\exists\, x^* < c$ $f(x^*) \geq L - \varepsilon$. Let $\delta = c - x^* > 0$

Suppose $\underbrace{c-\delta}_{x^*} < x < c$,

want: $|f(x) - L| < \varepsilon$

Note: Since $x < c$, $f(x) \leq L$

$L - f(x) < \varepsilon$

$L - \varepsilon \leq f(x^*) < f(x)$

$L - f(x) < \varepsilon$ ◡̈

(2) $f : \mathbb{R} \to \mathbb{R}$ diff., $c \in \mathbb{R}$ $\lim_{x \to c} f'(x) = L$

Prove $f'(c) = L$.

Method 1 By def. $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

By mean value thm, $\forall x \neq c$ $\exists p$ between c and x s.t. $\frac{f(x) - f(c)}{x - c} = f'(p)$

For each x, choose $p(x)$ as above

By the axiom of choice we have a function $p(x)$ s.t. $\forall x$ $p(x)$ is between c and x

Sandwich: $p \to c$ as $x \to c$

$\therefore f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{p \to c} f'(p) = L$

☺

Method 2: $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

As $x \to c$, $x - c \to 0$. Since f is diff., it is cont.

So $f(x) \to f(c)$, So $f(x) - f(c) \to 0$.

This is $\frac{0}{0}$ version of L'Hôpital.

$\frac{[f(x) - f(c)]'}{(x - c)'} = f'(x) \to L$ $\therefore \frac{f(x) - f(c)}{x - c} \to L$

☺

(3) $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

a) $f'(x) = ?$

If $x \neq 0$ $\quad f'(x) = [x^2]' \sin\left(\frac{1}{x}\right) + x^2 \left[\sin\left(\frac{1}{x}\right)\right]'$

$$= 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right) \underbrace{\left[\frac{1}{x}\right]'}_{-\frac{1}{x^2}}$$

$$\boxed{= 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)}$$

for $x \neq 0$.

$$f'(0) = \lim_{\substack{x \to 0 \\ (x \neq 0)}} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right) - 0}{x}$$

$$= \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

Sandwich: $0 \leq \left|x \sin\left(\frac{1}{x}\right)\right| \leq |x| \left|\sin\left(\frac{1}{x}\right)\right| \leq |x| \to 0$ as $x \to 0$

b) Suppose $\lim_{x \to c} f'(x) = L$

$$\parallel$$

$$\lim_{x \to c} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

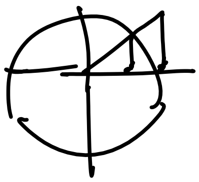
$$\cos\left(\frac{1}{x}\right) = 2x \sin\left(\frac{1}{x}\right) - f'(x)$$

$$\lim_{x \to 0} \cos\left(\frac{1}{x}\right) = \lim 2x \sin\left(\frac{1}{x}\right) - \lim f'(x)$$

$$= 0 - L = -L$$

Let $x_n = \frac{1}{n\pi}$, then $x_n \to 0$, but $\cos\left(\frac{1}{x_n}\right) = \cos(n\pi) = (-1)^n$ diverges ☹

(4) $\lim\limits_{x \to 0} \frac{\sin x}{x}$ $\qquad$ $\lim\limits_{x \to \infty} \frac{\sin x}{x} = 0$



$\frac{0}{0}$ l'Hôpital

$$\frac{(\sin x)'}{x'} = \cos x \to 1 \text{ as } x \to 0$$

sandwich

$$0 \leq \left| \frac{\sin x}{x} \right| = \frac{|\sin x|}{|x|} \leq \frac{1}{|x|} \to 0$$

(5) a) $f(x) = \cos(x)$

$f'(x) = -\sin(x)$

$f''(x) = -\cos(x)$

$f'''(x) = \sin(x)$

$f''''(x) = \cos x$

$f^{\cancel{'''''}}$ ... etc.

$$f^{(n)}(x) = \begin{cases} \cos x & \text{for } n \equiv 0 \bmod 4 \\ -\sin x & 1 \bmod 4 \\ -\cos x & 2 \bmod 4 \\ \sin x & 3 \bmod 4 \end{cases}$$

$$f^{(n)}(0) = \begin{cases} 1 & \text{for } n \equiv 0 \bmod 4 \\ 0 & 1 \bmod 4 \\ -1 & 2 \bmod 4 \\ 0 & 3 \bmod 4 \end{cases}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots c_n \frac{x^n}{n!}$$

where $c_n = 1, 0, -1, 0$ as above.

b) $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$ for some c between 0 and x

sin/cos

$$0 \le |R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} |x|^{n+1} \le \frac{|x|^{n+1}}{(n+1)!} \to 0$$