① a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -7x + 3y \\ -18x + 8y \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ -18 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$A$ $\quad \det A = -56 + 54 = -2 \neq 0$

$\therefore$ invertible

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is

the unique homog. sol.

$(A\bar{x} = \bar{0})$

$$\det(A - \lambda I) = \det \begin{bmatrix} -7-\lambda & 3 \\ -18 & 8-\lambda \end{bmatrix}$$

$$= (-7-\lambda)(8-\lambda) + 54 = \lambda^2 + 7\lambda - 8\lambda - 56 + 54$$

$$= \lambda^2 - \lambda - 2 \qquad \boxed{\lambda = 2, -1}$$

$2 > 0 \quad \therefore$ unstable

b) $A - 2I = \begin{bmatrix} -9 & 3 \\ -18 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$

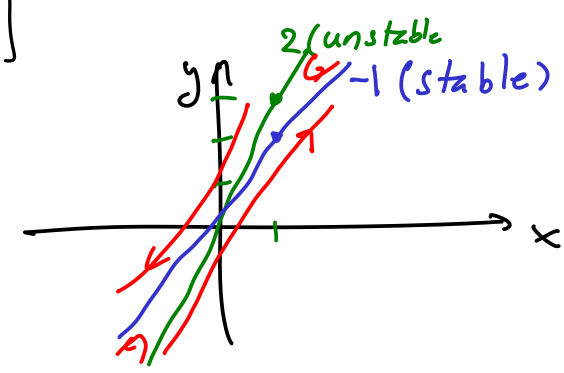
$x - \frac{1}{3}y = 0$

Let $\bar{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$A + I = \begin{bmatrix} -6 & 3 \\ -18 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$

$x - \frac{1}{2}y = 0$

Let $\bar{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



(2) $x'' = t^2 \qquad x(0) = x(2) = 0$

Let $x_1 = t \qquad x_2 = t-2$

$$W = \det \begin{bmatrix} x_1 & x_2 \\ x_1' & x_2' \end{bmatrix} = t - (t-2) = 2$$

$$x(t) = t \int_t^2 \frac{s^2(s-2)}{2}\,ds + (t-2)\int_0^t \frac{s^2 s}{2}\,ds$$

$$= \frac{t}{2}\int_t^2 \left(s^3 - s^2\right)ds + \frac{t-2}{2}\int_0^t s^3\,ds$$

$$= \frac{t}{2}\left[\frac{s^4}{4} - \frac{s^3}{3}\right]_t^2 + \frac{t-2}{2}\left[\frac{s^4}{4}\right]_0^t$$

$$= \frac{t}{2}\left[4 - \frac{8}{3} - \left(\frac{t^4}{4} - \frac{t^3}{3}\right)\right] + \frac{t-2}{2}\,\frac{t^4}{4}$$

$$= \cdots = \boxed{\frac{t^4 - 8t}{12}}$$

Check: $x(0) = x(2) = 0$

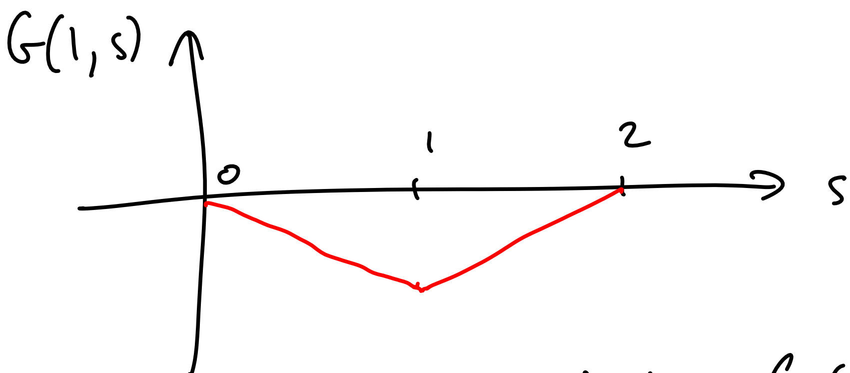
$$\left(\frac{t^4 - 8t}{12}\right)'' = \left(\frac{4t^3 - 8}{12}\right)' = \frac{12t^2}{12} = t^2 \quad \checkmark$$

Since 0 is the unique sol. to $x'' = 0 \quad x(0) = x(2) = 0$
this is the unique sol. to the BVP

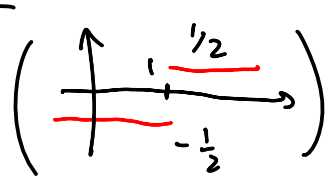
b) $$G(t,s) = \begin{cases} \dfrac{(t-2)s}{2} & \text{for } 0 \le s \le t \\ \dfrac{t(s-2)}{2} & \text{for } t \le s \le 2 \end{cases}$$

$$x(t) = \int_0^2 s^2 G(t,s)\,ds$$

$$G(1,s) = \begin{cases} -\dfrac{s}{2} & \text{for } 0 \le s \le 1 \\ \dfrac{s-2}{2} & \text{for } 1 \le s \le 2 \end{cases}$$




Check: $G(1,s)''$

$=$ (  )$'$



$\underbrace{\phantom{xxxx}}_{u(s-1)\ -\ \frac{1}{2}}$

$$= \left(u(s-1) - \tfrac{1}{2}\right)' = u'(s-1) = \delta(s-1)$$

3

$$x'' - x = t \qquad x(0) + x'(0) = 1$$
$$x(1) - x'(1) = 1$$

Char. eq. $\lambda^2 - 1 = 0 \qquad \lambda = 1, -1$

$$x_h = Ae^{t} + Be^{-t}$$

$$x_p = -t$$

$$x = Ae^{t} + Be^{-t} - t$$

$$x' = Ae^{t} - Be^{-t} - 1$$

$$x(0) + x'(0) = 1 \quad \Rightarrow \quad A + B + A - B - 1 = 1$$
$$2A = 2 \qquad A = 1$$

$$x(1) - x'(1) = 1$$

$$\cancel{Ae} + Be^{-1} \cancel{-1} - (\cancel{Ae} - Be^{-1} \cancel{-1}) = 1$$

$$2Be^{-1} = 1 \qquad B = \frac{e}{2}$$

$$\boxed{x(t) = e^{t} + \frac{1}{2}e^{1-t} - t}$$