1. $$x = t x' - (x')^2$$

$\frac{d}{dt}$
$$\cancel{x'} = \cancel{x'} + t x'' - 2x' x''$$

$$x''(t - 2x') = 0$$

$$x'' = 0 \quad \text{or} \quad t - 2x' = 0$$

$$x' = c \qquad\qquad x' = \frac{t}{2}$$

$$x = ct + k \qquad\qquad x = \frac{t^2}{4} + L$$
const.

Plug back in:

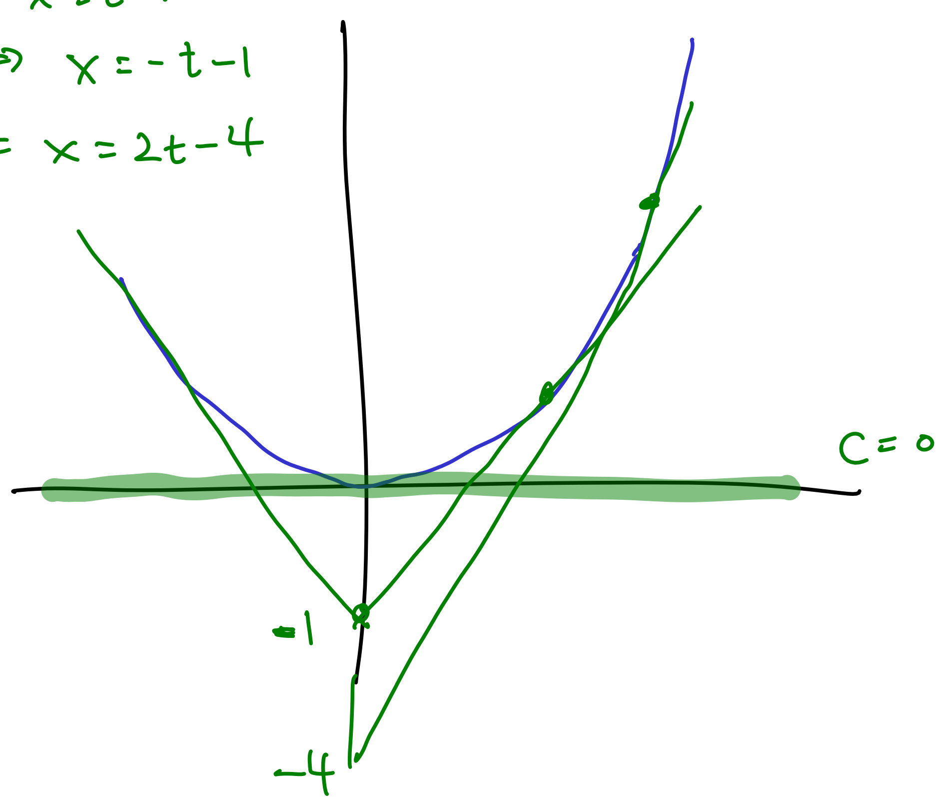
$$ct + k = tc - c^2$$
$$\therefore \ k = -c^2$$
$$\boxed{x = ct - c^2}$$

$$\frac{t^2}{4} + L = \underbrace{\frac{t^2}{2} - \frac{t^4}{4}}_{t^4/4}$$
$$\therefore L = 0$$
$$\boxed{x = \frac{t^2}{4}}$$

$c=1 \Rightarrow x = t-1$

$c=-1 \Rightarrow x = -t-1$

$c = 2 = x = 2t-4$



(2) $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$ $\quad \bar{x}' = A\bar{x} \quad$ $\bar{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ -2x-y \end{bmatrix}$$

Try $H_y = x+y \qquad H_x = 2x+y$

$(H_y)_x = 1 \qquad (H_x)_y = 1$ ☺

Partial integration

$H_x = 2x+y$

$H = x^2 + xy + h(y)$

$H_y = x + h'(y) = x+y$

$h'(y) = y$

$h(y) = \frac{y^2}{2} + C$

$$\boxed{H = x^2 + xy + \frac{y^2}{2} + C}$$

disc. $= B^2 - 4AC = 1 - 4 \cdot 1 \cdot \frac{1}{2} = 1 - 2 = -1 < 0$

$\therefore$ ellipses ☺ (as in Ex. 8.1.3)

3. $y'' = xy$, $y'' - xy = 0$

Expand at $x=0$ (ordinary point)

Try $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$

shift indices

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

Collect terms x^0 : $a_2 \cdot 2 \cdot 1 = 0$

x^n, $n \geq 1$: $a_{n+2}(n+2)(n+1) - a_{n-1} = 0$

shift $n \geq 0$ $\quad a_{n+3} = \frac{1}{(n+3)(n+2)} a_n$

$a_3 = \frac{1}{6} a_0$

$a_7 = \frac{1}{42} a_4 = \frac{1}{504} a_1$

$a_4 = \frac{1}{12} a_1$

$a_5 = \ldots \cdot a_2 = 0$

$a_6 = \frac{1}{30} a_3 = \frac{1}{180} a_0$

$$y = a_0\left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \ldots\right) + a_1\left(x + \frac{1}{12}x^4 + \ldots\right)$$

$+ \frac{1}{504}x^7$

No singularities, so expect ∞ radius of convergence.

4 a) $\underbrace{(t^2-t)}_{t(t-1)}x'' + tx' + 3x = 0$

$\therefore$ Sing: $t = 0, 1$

$$x'' + \underbrace{\frac{1}{t-1}}_{p}x' + \underbrace{\frac{3}{t(t-1)}}_{q}x = 0$$

$tp = \dfrac{t}{t-1}$ $\leftarrow$ an. @ $t=0$

$t^2 q = \dfrac{3t}{t-1}$

$\therefore$ $t=0$ is a regular singular pt.

$p_0 = 0$, $q_0 = 0$

Indicial eq: $r(r-1) + p_0 r + q_0$

$r(r-1) = 0$ $\qquad r = 0, 1$

@ $t=1$: $(t-1)p = 1$ $\leftarrow$ an. @ $t=1$

$(t-1)^2 q = \dfrac{3(t-1)}{t}$

$\therefore$ $t=1$ is a regular singular pt.

$p_0 = 1$, $q_0 = 0$

$r(r-1) + r = 0$, $r^2 = 0$, $r = 0$ (double root)

b) $\underbrace{(t^2-1)^2}_{(t-1)^2(t+1)^2} x'' - (t-1)x' + 7x = 0$

Sing. pts: $t = 1, -1$

$$x'' - \underbrace{\frac{1}{(t-1)(t+1)^2}}_{p} x' + \underbrace{\frac{7}{(t-1)^2(t+1)^2}}_{q} x = 0$$

@ $t=1$

$(t-1)p = -\dfrac{1}{(t+1)^2}$ ← an. @ $t=1$

$(t-1)^2 q = \dfrac{7}{(t+1)^2}$

$\therefore$ $t=1$ is a regular singular pt.

$p_0 = -\frac{1}{4}$, $q_0 = \frac{7}{4}$

$r(r-1) - \frac{1}{4}r + \frac{7}{4} = 0$

$r^2 - \frac{5}{4}r + \frac{7}{4} = 0 \xrightarrow{\text{quadratic formula}} r = \frac{5}{8} \pm i\frac{\sqrt{87}}{8}$

@ $t=-1$

$(t+1)p = -\dfrac{1}{(t-1)(t+1)}$ ← not an. @ $t=-1$

$\therefore$ $t=-1$ is an irregular singular pt.

:)