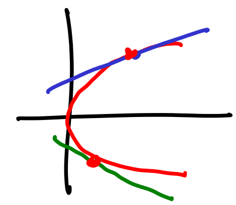
① $x = tx' + \frac{1}{x'}$, $\xrightarrow{d/dt}$ $\cancel{x'} = \cancel{x'} + tx'' - \frac{1}{(x')^2}x''$

$x''\left(t - \frac{1}{(x')^2}\right) = 0$, $x'' = 0$ or $t - \frac{1}{(x')^2} = 0$

If $x''=0$, $x'=c$, $x=ct+k$, $\cancel{ct}+k = t\cancel{c} + \frac{1}{c}$, $k=\frac{1}{c}$. $\boxed{x = ct + \frac{1}{c}}$

If $t - \frac{1}{(x')^2} = 0$ $(x')^2 = \frac{1}{t}$, $x' = \pm\frac{1}{\sqrt{t}}$, $x = \pm 2\sqrt{t} + L$

$\pm 2\sqrt{t} + L = \pm\frac{t}{\sqrt{t}} \pm \sqrt{t}$, so $L=0$, so $\boxed{x = \pm 2\sqrt{t}}$



② $x' = x - 4xy$ Let $H = x + 4y - 2\ln x - \ln y$

$y' = -2y + xy$ Then $\frac{dH}{dt} = \left(1 - \frac{2}{x}\right)x' + \left(4 - \frac{1}{y}\right)y' =$

$= \left(1-\frac{2}{x}\right)(x - 4xy) + \left(4 - \frac{1}{y}\right)(-2y + xy) = (x-2)(1-4y) + (4y-1)(-2+x) = 0$

$\therefore H(x,y) = k$ $x' = y' = 0 \Rightarrow x=0, y=0$ or $x=2, y=\frac{1}{4}$

$\text{Hessian}(H) = \begin{bmatrix} 2/x^2 & 0 \\ 0 & 1/y^2 \end{bmatrix}$ so H is concave up in a nbd. of $[2, \frac{1}{4}]$

so level curves of H that start near $[2, \frac{1}{4}]$ are periodic.

③ see solutions to Midterm 1.

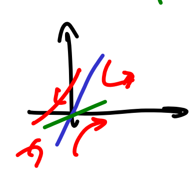
④ $A = \begin{bmatrix} -9 & 8 \\ -12 & 11 \end{bmatrix}$ $\det(A - \lambda I) = \det\begin{bmatrix} -9-\lambda & 8 \\ -12 & 11-\lambda \end{bmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$

$\lambda = 3, -1$

$\text{rref}(A - 3I) = \text{rref}\begin{bmatrix} -12 & 8 \\ -12 & 8 \end{bmatrix} = \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix}$ $x - \frac{2}{3}y = 0 \leftarrow$ unstable manifold

$\text{rref}(A + I) = \text{rref}\begin{bmatrix} -8 & 8 \\ -12 & 12 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x - y = 0 \leftarrow$ stable manifold

Since A is invertible $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the unique sol. to $x' = y' = 0$.



⑤ $x_h(t) = mt + b$, $x_p = t^2 - \frac{t^3}{2}$, $x = mt + b + t^2 - \frac{t^3}{2}$

$x(0) = 0 \Rightarrow b = 0$, $x(1) - x'(1) = 0 \Rightarrow \cancel{m} + 1 - \frac{1}{2} - \cancel{m} - 2 + \frac{3}{2} = 0$

$\boxed{x = mt + t^2 - \frac{t^3}{2}}$

⑥ $f(t) = 1 - t^2$. $a_0 = \int_0^1 (1-t^2)\,dt = \left[t - \frac{t^3}{3}\right]_0^1 = \frac{2}{3}$

Since f is even all $b_n = 0$. For $n \geq 1$ $a_n = 2\int_0^1 (1-t^2)\cos(n\pi t)\,dt$

$1-t^2 \quad + \quad \cos(n\pi t)$

$-2t \quad - \quad \frac{1}{n\pi}\sin(n\pi t)$

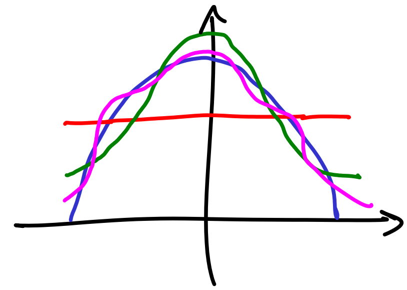
$-2 \quad + \quad -\frac{1}{n^2\pi^2}\cos(n\pi t)$

$0 \quad\quad -\frac{1}{n^3\pi^3}\sin(n\pi t)$

$$= 2\left[\frac{1-t^2}{n\pi}\sin(n\pi t) - \frac{2t}{n^2\pi^2}\cos(n\pi t) + \frac{2}{n^3\pi^3}\sin(n\pi t)\right]_0^1$$

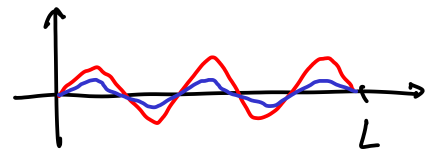
$$= -\frac{4}{n^2\pi^2}(-1)^n$$

$$f(t) \approx \frac{2}{3} + \frac{4}{\pi^2}\cos(\pi x) - \frac{1}{\pi^2}\cos(2\pi x)$$



⑦ $u = \sin\left(\frac{5\pi x}{L}\right)\cos\left(c\,\frac{5\pi}{L}t\right)$

(see midterm 3 for details)



⑧ $u = 25 - \frac{1}{3}\sin(2\theta)\,r^2$