old Final, 2

a) $x^3 y'' + \alpha x y' + \beta y = 0$

$\alpha \neq 0.$

$$y'' + \frac{\alpha x}{x^3} y' + \frac{\beta}{x^3} y = 0$$

$$y'' + \underbrace{\frac{\alpha}{x^2}}_{p} y' + \frac{\beta}{x^3} y = 0$$

$\neq 0$

$xp = \frac{\alpha}{x}$ ← not analytic at $x=0$

$\therefore$ $x=0$ is an irreg. sing. pt.

b) $x(1-x)y'' + [\gamma - (1+\alpha+\beta)x]y' - \alpha\beta y = 0$

Singularities: $x = 0, 1$

$$y'' - \underbrace{\frac{\gamma-(1+\alpha+\beta)x}{x(1-x)}}_{p} y' - \underbrace{\frac{\alpha\beta}{x(1-x)}}_{q} y = 0$$

$xp = -\frac{\gamma - (1+\alpha+\beta)x}{1-x}$ ← analytic @ $x=0$:)

$x^2 q = -\frac{\alpha\beta x}{1-x}$ ← ditto

$\therefore$ $x=0$ is a regular singularity

$(x-1)p = \frac{\gamma-(1+\alpha+\beta)x}{x}$ ← analytic @ $x=1$

$(x-1)^2 q = \frac{\alpha\beta(x-1)}{x}$ ← ditto

$\therefore$ $x=1$ is a regular singularity.

Indicial eq.

@ $x=0$

$$xp = -\frac{\gamma-(1+\alpha+\beta)x}{1-x}$$

$$x^2q = -\frac{\alpha\beta x}{1-x}$$

plug in $x=0$

$-\gamma$

0

$$xp = -\left(\gamma-(1+\alpha+\beta)x\right)\left(1+x+x^2+\dots\right) = \underbrace{-\gamma}_{p_0} + \dots$$

$$x^2q = -\alpha\beta x\left(1+x+x^2+\dots\right) = \underbrace{0}_{q_0} + \dots$$

Indicial eq.: $\underline{r(r-1)+p_0 r+q_0 = 0}$

$$r^2-r-\gamma r = 0$$

$$r(r-1-\gamma)=0$$

$$\underline{r=0,\ 1+\gamma}$$

@ $x=1$

$$(x-1)p = \frac{\gamma-(1+\alpha+\beta)[(x-1)+1]}{(x-1)+1}$$

$$= \left(\gamma-(1+\alpha+\beta)[(x-1)+1]\right)\left(1-(x-1)+(x-1)^2\dots\right)$$

plug in $x=1$

$$(x-1)p = \frac{\gamma-(1+\alpha+\beta)x}{x}$$

$$(x-1)^2q = \frac{\alpha\beta(x-1)}{x}$$

0

$$\therefore\ p_0 = \gamma-1-\alpha-\beta$$

:)

$$q_0 = 0$$

$$\underline{r(r-1)+(\gamma-1-\alpha-\beta)r=0}$$

$$r(r+\gamma-2-\alpha-\beta)=0 \qquad \underline{r=0,\ -\gamma+2+\alpha+\beta}$$

3. $y'' = xy$, $y'' - xy = 0$

Expand at $x=0$ (ordinary point)

Try $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$

shift indices

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

Collect terms x^0 : $a_2 \cdot 2 \cdot 1 = 0$

x^n, $n \geq 1$: $a_{n+2}(n+2)(n+1) - a_{n-1} = 0$

shift

$n \geq 0$ $\quad a_{n+3} = \frac{1}{(n+3)(n+2)} a_n$

$a_3 = \frac{1}{6} a_0$

$a_4 = \frac{1}{12} a_1$

$a_5 = \dots \cdot a_2 = 0$

$a_6 = \frac{1}{30} a_3 = \frac{1}{180} a_0$

$$y = a_0\left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \dots\right) + a_1\left(x + \frac{1}{12}x^4 + \dots\right)$$