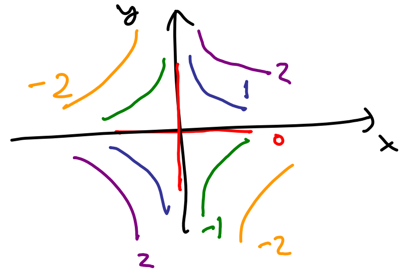
Midterm 1 / 2008.10.16 / Calculus for Applications / MAT 3243.001

1. Sketch and label 5 level sets of $f(x, y) = xy$, including one at level 0.

$xy = 0 \Rightarrow x = 0$ or $y = 0$

$xy = \pm 1 \Rightarrow y = \pm \frac{1}{x}$

$xy = \pm 2 \Rightarrow y = \pm \frac{2}{x}$



2. In each case determine whether the limit exists, and if so, find the limit.

$$\text{(a)} \lim_{[x,y]\to 0} \frac{x^4 - y^4}{x^2 + y^2} \qquad \text{(b)} \lim_{[x,y]\to 0} \frac{x^2 - y^2}{x^2 + y^2}$$

a) $\frac{x^4 - y^4}{x^2+y^2} = \frac{(x^2-y^2)\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}$ if $[x,y] \neq [0,0]$ $= x^2 - y^2 \to 0$ as $[x,y] \to [0,0]$

b) If $x = 0$, we get $\frac{-y^2}{y^2} \to -1$

If $y = 0$, we get $\frac{x^2}{x^2} \to 1$

$\neq$ $\therefore$ Limit does not exist

3. If a cucaracha crawls south at 1 cm/s, it notices an increase in temperature $T(x,y)$ at the rate of 2°/s. If it crawls east at 1 cm/s, the temperature increases by 4°/s. What is the rate of change of temperature if the cucaracha crawls northeast at 2 cm/s?

Clue: $\frac{\text{deg}}{\text{cm}} \cdot \frac{\text{cm}}{\text{s}} = \frac{\text{deg}}{\text{s}}$

$\therefore T_x = 4 \qquad T_y = -2$

NE : $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad D_{NE} T = \text{grad}\, T \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} [4, -2] \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \frac{1}{\sqrt{2}} (4-2) = \sqrt{2} \approx 1.41$

$\frac{\text{deg}}{\text{s}} = \sqrt{2} \cdot 2 \approx 2.82$

$\therefore$ the temperature increases by 2.82°/s

4. Find the divergence and curl of $F = [y^2 z, \exp(xyz), x^2 y]$.

$\text{div}\, F = F_{1x} + F_{2y} + F_{3z} = 0 + e^{xyz} xz + 0 = xze^{xyz}$

$\text{curl}\, F = [F_{3y} - F_{2z}, F_{1z} - F_{3x}, F_{2x} - F_{1y}]$

$= [x^2 - e^{xyz} xy, y^2 - 2xy, e^{xyz} yz - 2yz]$

5. Let $f = (1+x^2+y^2)^{-1}$. Compute the Hessian matrix for f and find the quadratic Taylor approximation to f at the origin.

$f_x = -(1+x^2+y^2)^{-2}2x \quad f_y = -(1+x^2+y^2)^{-2}2y$

$f_{xx} = 2(1+x^2+y^2)^{-3}2x \cdot 2x - (1+x^2+y^2)^{-2}2$

$f_{xy} = f_{yx} = 2(1+x^2+y^2)^{-3} \cdot 2y \cdot 2x$

$f_{yy} = 2(1+x^2+y^2)^{-3}2y2y - (1+x^2+y^2)^{-2}2$

Eval: $f(0,0)=1$, $Df(0,0) = [0,0]$, $Hf(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

$\therefore\ f(x,y) \approx 1 + [0,0]\begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2}(x\ y)\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \boxed{1 - x^2 - y^2}$

Check: Geom. Ser.: $\frac{1}{1-z} = 1 + z + \ldots$ Subs $z = -x^2 - y^2$: $\frac{1}{1+x^2+y^2} = 1 - x^2 - y^2 \ldots$