Midterm 2 / 2007.11.27 / Calculus for Applications / MAT 3243.001

Note Title 11/27/2007

1. Use cylindrical coordinates to integrate $(x^2+y^2+3z^2)\,dx\,dy\,dz$ over the solid $\underbrace{x^2+y^2}_{r^2} \le 4$, $-2 \le z \le 1$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix} \qquad \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} dr\cos\theta - r\sin\theta\, d\theta \\ dr\sin\theta + r\cos\theta\, d\theta \\ dz \end{bmatrix}$$

$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

$$\int (x^2+y^2+3z^2)\,dx\,dy\,dz = \int_{-2}^{1}\left[\int_{-\pi}^{\pi}\left[\int_0^2 \underbrace{(r^2+3z^2)\,r}_{r^3+3z^2 r}\,dr\right]d\theta\right]dz$$

$$= \int_{-2}^{1}\left[\int_{-\pi}^{\pi}\left(\frac{r^4}{4}+3z^2\frac{r^2}{2}\right)\Bigg|_{r=0}^{r=2} d\theta\right]dz = \int_{-2}^{1}\left[\int_{-\pi}^{\pi}\left[4+6z^2\right]d\theta\right]dz$$

$$= 2\pi\cdot 2\int_{-2}^{1}(2+3z^2)\,dz = 4\pi\left[2z+z^3\right]\Big|_{-2}^{1} = 4\pi\underbrace{(2+1-(-4)-(-8))}_{15} = \boxed{60\pi}$$

2. Either find a scalar potential for F or explain why it fails to exist, where

(a) $F = [y, -x, 0]$ (b) $F = [x, y, z]$

a) $$\operatorname{curl} F = \det\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = [0, 0, -2] \ne 0$$

Since curl(grad) = 0, No potential exists.

b) curl $F = 0$, so a potential exists.

By inspection it is $\boxed{\frac{1}{2}(x^2+y^2+z^2) + C}$

3. Either find a vector potential for F or explain why it fails to exist, where

(a) $F = [2x, -y, -z]$ (b) $F = [3x, -y, -z]$

a) div $F = 0$, so a vector potential exists

Let $\omega = A\,dx + B\,dy$, then $d\omega = (A_x\,dx + A_y\,dy + A_z\,dz)\,dx$
$+ (B_x\,dx + B_y\,dy + B_z\,dz)\,dy$

$$= -B_z\,dy\,dz + A_z\,dz\,dx + (B_x - A_y)\,dx\,dy$$

Since we want $-B_z = 2x$, let $B = -2xz$

and since we want $A_z = -y$, let $A = -yz$

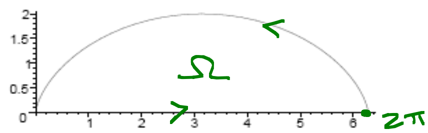
then $d\omega = 2x\,dy\,dz - y\,dz\,dx + (\underbrace{-2z+z}_{-z})\,dx\,dy$

$\therefore$ we can use $\boxed{[-yz, -2xz, 0]}$

check: $\det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yz & -2xz & 0 \end{pmatrix} = [\,2x, -y, \underbrace{-2z+z}_{-z}\,]$

b) $\operatorname{div} F = 3-1-1 = 1 \neq 0$ so since $\operatorname{div}(\operatorname{curl}) = 0$, no vector potential exists

4. Use Green's theorem to calculate the area under one arch of the cycloid $[x, y] = [t - \sin t, 1 - \cos t]$ pictured below.



Hint: Find a 1-form ω such that $d\omega = dx\,dy$ and recall that $\int_\Omega d\omega = \int_{\partial\Omega} \omega$

Let $\omega = x\,dy$, then $d\omega = dx\,dy$

$$\text{Area} = \int_\Omega dx\,dy = \int_\Omega d\omega = \int_{\partial\Omega} \omega = \int_{\partial\Omega} x\,dy$$

Parametrize $\partial\Omega$:

Part I $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ $\quad 0 \le x \le 2\pi$

Since $dy = 0$, we get 0.

Part II $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t - \sin t \\ 1 - \cos t \end{bmatrix}$ $\quad 2\pi \xrightarrow{t} 0$ $\quad dy = \sin t\,dt$

$$\int x\,dy = \int_{2\pi}^{0} (t - \sin t)\sin t\,dt = \int_{2\pi}^{0} [t\sin t - (\sin t)^2]\,dt$$

t	$\sin t$
1	$-\cos t$
0	$-\sin t$

$(\sin t)^2 = \frac{1}{2} - \frac{1}{2}\cos(2t)$

$\downarrow$ antider.

$\frac{1}{2}t - \frac{1}{4}\sin(2t)$

$$\left[-t\cos t + \sin t - \frac{1}{2}t + \frac{1}{4}\sin(2t)\right]_{2\pi}^{0} = 2\pi + \pi = \boxed{3\pi}$$

5. Find the flux of $[7x, 8y, 9z]$ through the unit sphere.

Hint: Don't do it directly. ◡̈

Let Ω = the unit Ball. Then $\partial\Omega$ = the unit sphere

$$\int_{\partial\Omega} [7x, 8y, 9z] \cdot d\bar{S} = \int_{\Omega} \operatorname{div} [7x, 8y, 9z] \, dV$$

$$= \int_{\Omega} (7+8+9) \, dV = 24 \int_{\Omega} dV = 24 \cdot \operatorname{vol}(\Omega) = 24 \cdot \tfrac{4}{3}\pi = \boxed{32\pi}$$