1. Let $\mathbf{r} = [x, y, z]$ and $r = |\mathbf{r}|$. Express $\nabla \cdot (r^n \mathbf{r})$ in terms of r.

The x component of $r^n \bar{r}$ is $r^n x = (x^2+y^2+z^2)^{n/2} x$

$$\frac{\partial}{\partial x}(r^n x) = \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} 2x \cdot x + (x^2+y^2+z^2)^{n/2} \cdot 1 \quad \leftarrow \frac{n-2}{2}$$

$$= n r^{n-2} x^2 + r^n$$

For the other components, similarly, $\frac{\partial}{\partial y}(r^n y) = n r^{n-2} y^2 + r^n$

$$\frac{\partial}{\partial z}(r^n z) = n r^{n-2} z^2 + r^n$$

$$\therefore \nabla \cdot (r^n \bar{r}) = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z) = n r^{n-2} \underbrace{(x^2+y^2+z^2)}_{r^2} + 3r^n$$

$$= n r^n + 3r^n = \boxed{(n+3) r^n}$$

2. Let $\omega = x\,dx + y\,dy + z\,dz$ and $\eta = (x^2+yz)\,dy\,dz + (y^2+zx)\,dz\,dx + (z^2+xy)\,dx\,dy$. Compute $d\eta$ and $\omega \wedge \eta$.

$$d\eta = (2x\,dx + dy \cdot z + y\,dz)\,dy\,dz + (2y\,dy + dz \cdot x + z\,dx)\,dz\,dx$$
$$+ (2z\,dz + dx \cdot y + x\,dy)\,dx\,dy$$

$$= 2x\,dx\,dy\,dz + 2y\,dy\,dz\,dx + 2z\,dz\,dx\,dy$$

(⊕ ⊕)

$$= \boxed{2(x+y+z)\,dx\,dy\,dz}$$

$$\omega \wedge \eta = (x\,dx + y\,dy + z\,dz) \wedge [(x^2+yz)\,dy\,dz + (y^2+zx)\,dz\,dx + (z^2+xy)\,dx\,dy$$

$$= x(x^2+yz)\,dx\,dy\,dz + y(y^2+zx)\,dy\,dz\,dx + z(z^2+xy)\,dz\,dx\,dy$$

(⊕ ⊕)

$$= \boxed{(x^3+y^3+z^3+3xyz)\,dx\,dy\,dz}$$

3. Given a steady temperature distribution $f(x,y) = x^y$, how quickly does the temperature change as you start moving from the point $[3,2]$ towards $[2,3]$ with speed 5?

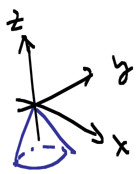
grad $f = [y x^{y-1},\ x^y \ln x]$ eval @ $[3,2]$: $[6,\ 9\ln 3]$

direction vector: $\begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ Normalize: $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

directional derivative: $[6,\ 9\ln 3] \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(9\ln 3 - 6) = \frac{3}{\sqrt{2}}(3\ln 3 - 2)$

∴ the temperature change at the rate is $\boxed{\frac{15}{\sqrt{2}}(3\ln 3 - 2)}$

4. Use cylindrical coordinates to parametrize the solid cone $z^2 = x^2 + y^2$, $-1 \le z \le 0$. Integrate $(x^2 + y^2 - z^2)\,dx\,dy\,dz$ over this cone.



Since $z \le 0$, $z = -\sqrt{x^2+y^2} = -r$ on the lateral surface

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix} \quad \begin{matrix} 0 \le r \le 1 \\ -\pi < \theta \le \pi \\ -1 \le z \le -r \end{matrix} \quad \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} dr\cos\theta - r\sin\theta\, d\theta \\ dr\sin\theta + r\cos\theta\, d\theta \\ dz \end{bmatrix}$$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

convert to iterated

$$\int (x^2+y^2-z^2)\,dx\,dy\,dz = \int (r^2 - z^2)\, r\,dr\,d\theta\,dz = \int_{-\pi}^{\pi}\left[\int_0^1 \left[\int_{-1}^{-r} (r^3 - rz^2)\,dz\right] dr\right] d\theta$$

$$= \int_0^1 \left[r^3 z - r\frac{z^3}{3}\right]_{-1}^{-r} dr \cdot \underbrace{\int_{-\pi}^{\pi} d\theta}_{2\pi} = 2\pi \int_0^1 \Big[\underbrace{-r^4 + \frac{r^4}{3} - \left(-r^3 + \frac{r}{3}\right)}_{-\frac{2}{3}r^4 + r^3 - \frac{r}{3}}\Big]\, dr$$

$$= 2\pi\left[-\frac{2}{15}r^5 + \frac{r^4}{4} - \frac{r^2}{6}\right]_0^1 = 2\pi\left(-\frac{2}{15} + \frac{1}{4} - \frac{1}{6}\right) = \frac{\pi}{30}(\underbrace{-8 + 15 - 10}_{-3})$$

$$= \boxed{-\frac{\pi}{10}}$$

5. Either find a scalar potential for $[3x^2, z^2/y, 2z\ln y]$ or explain why it fails to exist.

$$\text{curl} = \det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2 & z^2/y & 2z\ln y \end{bmatrix} = \left[\frac{2z}{y} - \frac{2z}{y}, 0, 0\right] = 0$$

Since the domain (the half-plane $y>0$) is simply connected, there exists a scalar u s.t. grad u is our vector field,

i.e. $u_x = 3x^2$, $u_y = \frac{z^2}{y}$, $u_z = 2z\ln y$

$u = x^3 + f(y,z)$

$u_y = f_y = \frac{z^2}{y}$ so $f = z^2\ln y + g(z)$, so $u = x^3 + z^2\ln y + g$

$u_z = 2z\ln y + g_z \Rightarrow g_z = 0$, so g is const, so $u = x^3 + z^2\ln y + c$

6. Either find a vector potential for $[xy^2z, -y^3z, x^2y + y^2z^2]$ or explain why it fails to exist.

$\text{div} = y^2z - 3y^2z + y^2 2z = 0$ and the domain $\mathbb{R}^3$ is contractible so there exists a vector potential whose curl is our vector field.

To find it, first try $[A, B, 0]$.

$$\text{curl}[A,B,0] = \det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A & B & 0 \end{bmatrix} = [-B_z, A_z, B_x - A_y]$$

Since $-B_z$ should be xy^2z, Try $B = -\frac{1}{2}xy^2z^2$

and since $A_z = -y^3z$, try $A = -\frac{1}{2}y^3z^2$

Then $B_x - A_y = -\frac{1}{2}y^2z^2 + \frac{3}{2}y^2z^2 = y^2z^2$ almost!

All we need now is something whose curl is the missing $[0,0,x^2y]$

By inspection $\left[0, \frac{x^3}{3}y, 0\right]$ works: $\det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & \frac{x^3y}{3} & 0 \end{bmatrix} = [0,0,x^2y]$

$\therefore$ the following vector potential works:

$$[A,B,0] + \left[0, \frac{x^3}{3}y, 0\right] = \left[-\frac{1}{2}y^3z^2, -\frac{1}{2}xy^2z^2 + \frac{x^3}{3}y, 0\right]$$

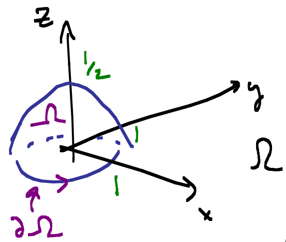
(not unique)

OR $\int_0^1 t\,[xy^2zt^4, -y^3zt^4, x^2yt^3 + y^2z^2t^4] \times [x,y,z]\,dt = \ldots$

$$= \left[-\frac{y^3z^3}{3} - \frac{x^2y^2}{5}, \frac{x^3y}{5}, \frac{xy^3z}{3}\right]$$

7. Verify the fundamental theorem $\int_\Omega d\omega = \int_{\partial\Omega} \omega$ with $\omega = xz\,dx + yz\,dy + (x^2+y^2)dz$ and the surface Ω given by $x^2+y^2+2z = 1$, $z \geq 0$ oriented with the upward normal. Sketch.

$z = \frac{1}{2}[1-(x^2+y^2)] = \frac{1}{2}(1-r^2)$



$$\Omega: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ \frac{1}{2}(1-r^2) \end{bmatrix} \qquad \begin{matrix} 0 \leq r \leq 1 \\ -\pi < \theta \leq \pi \end{matrix}$$

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} dr\cos\theta - r\sin\theta\,d\theta \\ dr\sin\theta + r\cos\theta\,d\theta \\ -r\,dr \end{bmatrix}$$

$$d\omega = (dx\cdot z + x\,dz)dx + (dy\cdot z + y\,dz)\,dy + (2x\,dx + 2y\,dy)\,dz$$

$$= x\,dz\,dx + \underbrace{y\,dz\,dy}_{-y\,dy\,dz} + \underbrace{2x\,dx\,dz}_{-2x\,dz\,dx} + 2y\,dy\,dz = y\,dy\,dz - x\,dz\,dx$$

$$\int_\Omega d\omega = \int \underset{r\sin\theta}{y}\,\underbrace{dy\,dz}_{-r^2\cos\theta\,d\theta\,dr} - \underset{r\cos\theta}{x}\,\underbrace{dz\,dx}_{r^2\sin\theta\,dr\,d\theta} = \int (r^3\sin\theta\cos\theta - r^3\sin\theta\cos\theta)\,dr\,d\theta$$

$$\boxed{= 0}$$

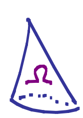
To parametrize $\partial\Omega$ set $z=0$. Then $r=1$, so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$ $-\pi < \theta \leq \pi$

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} d\theta, \text{ so } \int_{\partial\Omega} \omega = \int x\underset{0}{z}\,dx + y\underset{0}{z}\,dy + (x^2+y^2)\underset{0}{dz} \boxed{= 0}$$

Extra credit: Who first discovered the special case of the fundamental theorem that applies here?

The Stokes theorem was discovered by William Thomson (Lord Kelvin)

8. Let F be a smooth vector field on $\mathbb{R}^3$ such that the flux of F through the lateral surface of a cone of volume b is q. If F has constant divergence c, what is the flux of F through the base of the cone? Explain.



$\partial\Omega$ has two parts: L (lateral) and B (base)

$$\int_\Omega \operatorname{div} F\,dV = \int_{\partial\Omega} F\cdot dS = \underbrace{\int_L F\cdot dS}_{q} + \int_B F\cdot dS$$

$$\int_\Omega c\,dV = c\int_\Omega dV = cb$$

$$\therefore \int_B F\cdot dS = \boxed{cb - q}$$

Have a great break!