Calculus For Applications
MAT 3243.001 Final
Fall 2005

① a) $\pi^y \ln z + x\cos(xz) = 0$

$\downarrow d$

$$\ln\pi\ \pi^y \ln z\, dy + \pi^y \frac{1}{z} dz + \cos(xz)\, dx$$
$$+ x(-\sin(xz))(z\,dx + x\,dz) = 0$$

$\downarrow$ collect

$$dx[\cos(xz) - xz\sin(xz)] + dy[\ln\pi\ \pi^y \ln z]$$
$$+ dz\left[\frac{\pi^y}{z} - x^2\sin(xz)\right] = 0$$

$\downarrow$ eval. @ $[\pi, 1, 1]$

$$dx[-1-0] + dy[0] + dz[\pi - 0] = 0$$

Tangent plane: $-(x-\pi) + \pi(z-1) = 0$

$$\boxed{-x + \pi z = 0}$$

b) $s = \begin{bmatrix} \cos(\pi t) \\ \sin(\pi t) \\ 2t \end{bmatrix}$ $\quad ds = \begin{bmatrix} -\pi\sin(\pi t) \\ \pi\cos(\pi t) \\ 2 \end{bmatrix} dt$

$s(t) = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \Rightarrow 2t = 3 \Rightarrow t = \frac{3}{2}$

Tangent line: $s(\frac{3}{2}) + s'(\frac{3}{2})(t - \frac{3}{2})$

$$= \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} \pi \\ 0 \\ 2 \end{bmatrix}(t - \tfrac{3}{2}) = \boxed{\begin{bmatrix} \pi t - \frac{3\pi}{2} \\ -1 \\ 2t \end{bmatrix}}$$

② a) $F = \begin{bmatrix} x + yz \\ y + zx \\ z + xy \end{bmatrix}$ $DF = \begin{bmatrix} 1 & z & y \\ z & 1 & x \\ y & x & 1 \end{bmatrix}$

$\nabla \cdot F = \operatorname{tr} DF = 3$

b) $u = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + xyz + C$

$\nabla \times F = \nabla \times (\nabla u) = 0$ (automatically)

③

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2y + z \\ 3z + x \\ x + 2y \end{bmatrix}$$

call this g

Chain rule: $Df\, Dg$

$Df = [f_u \; f_v \; f_w]$

$Dg = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

$\therefore [f_x \; f_y \; f_z] = [f_u \; f_v \; f_w] \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

$\boxed{= [f_v + f_w, \; 2f_u + 2f_w, \; f_u + 3f_v]}$

④ $f = x + 2y + \ln(xy)$

$df = dx + 2dy + \frac{1}{xy}(y\,dx + x\,dy)$

$= dx\left[1 + \frac{1}{x}\right] + dy\left[2 + \frac{1}{y}\right]$

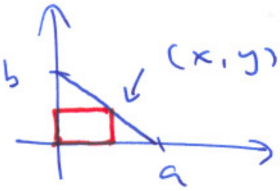
$df = 0 \Rightarrow x = -1, \; y = -\frac{1}{2}$ $\therefore$ Critical pts $= \{[-1, -\frac{1}{2}]\}$

$Hf = D\begin{bmatrix} 1 + \frac{1}{x} \\ 2 + \frac{1}{y} \end{bmatrix} = \begin{bmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{bmatrix}$ $\det Hf = \frac{1}{x^2 y^2}$

$\det Hf\left(\begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}\right) > 0$ $\therefore$ max or min

Since $-\frac{1}{x^2}$ @ $\begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix} < 0$ $[-1, -\frac{1}{2}]$ is a local max

(5)



Objective $f = xy$ (area)

Constraint: $g = \frac{x}{a} + \frac{y}{b} - 1$

Let $\mathcal{L} = f - \lambda g = xy - \lambda\left(\frac{x}{a} + \frac{y}{b} - 1\right)$

$\text{grad}\, \mathcal{L} = \left[y - \frac{\lambda}{a},\ x - \frac{\lambda}{b},\ 1 - \frac{x}{a} - \frac{y}{b}\right]$

$\text{grad}\, \mathcal{L} = 0 \Rightarrow y = \frac{\lambda}{a},\ x = \frac{\lambda}{b}$ so $\frac{y}{x} = \frac{b}{a}$, $y = \frac{b}{a}x$

$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{x}{a} + \frac{\frac{b}{a}x}{b} = 1 \Rightarrow \frac{2x}{a} = 1 \Rightarrow x = \frac{a}{2}$

$\Rightarrow y = \frac{b}{a} \cdot \frac{a}{2} = \frac{b}{2}$ $\therefore$ optimal rectangle has sides $\frac{a}{2}$ & $\frac{b}{2}$

(6) $\eta = y\,dx + x\,dy + z\,dz$

$d\eta = dy\,dx + dx\,dy + dz\,dz = 0$ $\therefore$ the integral is path indep.

In fact $\eta = d\left(xy + \frac{z^2}{2}\right)$, so $\int_{[0,1,1]}^{[1,2,3]} \eta = xy + \frac{z^2}{2}\Big|_{[0,1,1]}^{[1,2,3]}$

$= 2 + \frac{9}{2} - \left[0 + \frac{1}{2}\right] = 2 + \frac{8}{2} = \boxed{6}$

(7) $X = \begin{bmatrix} s^2t \\ st^2 \\ st \end{bmatrix}$ $DX = \begin{bmatrix} 2st & s^2 \\ t^2 & 2st \\ t & s \end{bmatrix}$

$X = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $DX\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -4 & 1 \\ 4 & -4 \\ -2 & 1 \end{bmatrix}$

Tangent plane: $X\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) + DX\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) \cdot \begin{bmatrix} s-1 \\ t+2 \end{bmatrix}$

$= \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} -4 & 1 \\ 4 & -4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} s-1 \\ t+2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} -4 \\ 4 \\ -2 \end{bmatrix}(s-1) + \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}(t+2)$

$= \boxed{\begin{bmatrix} -4s + t + 4 \\ 4s - 4t - 8 \\ -2s + t + 2 \end{bmatrix}}$

$N = \begin{bmatrix} -4 \\ 4 \\ -2 \end{bmatrix} \times \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 12 \end{bmatrix} = 2\begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}$

Eq: $-2(x+2) + y - 4 + 6(z+2) = 0$

$-2x + y + 6z = -4$, $\boxed{2x - y - 6z = 4}$

(8) Disc: $d\bar{S} = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} dr\, d\theta$

$$\bar{F} \cdot d\bar{S} = 2 \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} dr\, d\theta$$

$$\int \bar{F} \cdot d\bar{S} = 2 \text{ Area (disc)} = \boxed{2\pi}$$

$$\text{div } \bar{F} = 1 + 1 - 1 = 1$$

$$\iiint \text{div } F \, dV = \text{Volume (half ball)} = \frac{1}{2} \frac{4}{3} \pi = \frac{2\pi}{3}$$

$$\parallel$$

$$\iint_{\text{hemisphere}} \bar{F} \cdot d\bar{S} - 2\pi$$

$$\therefore \iint_{\text{hemisphere}} \bar{F} \cdot d\bar{S} = \frac{2\pi}{3} + 2\pi = 2\pi \cdot \left(\frac{1}{3} + 1\right) = \boxed{\frac{8\pi}{3}}$$

Have a great Break!

(See you next year)

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