Midterm 1 / 2004.10.13 / Calculus for Applications / MAT 3243

Note Title 10/14/2008

1. A surface in $\mathbf{R}^3$ is given by $e^{xy} + e^{xz} - 2e^{yz} = 0$. Find an equation for the plane tangent to this surface at $(-1,-1,-1)$.

Take d:
$$e^{xy}(dx\, y + x\, dy) + e^{xz}(dx\, z + x\, dz) - 2e^{yz}(dy\, z + y\, dz) = 0$$

$$(e^{xy} y + e^{xz} z)\, dx + (e^{xy} x - 2e^{yz} z)\, dy + (e^{xz} x - 2e^{yz} y)\, dz = 0$$

Eval:
$$-2e\, dx + e\, dy + e\, dz = 0$$

$$\boxed{-2(x+1) + (y+1) + (z+1) = 0}$$

$$-2x + y + z = 0$$

2. Find a parametric formula for the line tangent to the path $(5\cos(3t), 6t, 5\sin(3t))$ at the point $(5,0,0)$.

Direction vector:
$$\begin{bmatrix} 5\cos(3t) \\ 6t \\ 5\sin(3t) \end{bmatrix}' = \begin{bmatrix} -5\sin(3t)\cdot 3 \\ 6 \\ 5\cos(3t)\cdot 3 \end{bmatrix}$$

$$= 3\begin{bmatrix} -5\sin(3t) \\ 2 \\ 5\cos(3t) \end{bmatrix}$$

Eval @ $t=?$ Since $6t=0$ when $t=0$

Eval @ $t=0$

$$v = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \quad \text{(don't need 3)}$$

$$u + sv = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 2s \\ 5s \end{bmatrix}$$

scalar field   Vector field

3. Let $f(x, y, z) = x^2 z$ and $F(x, y, z) = (0, e^{xyz}, 0)$.

(a) Compute the directional derivative of f along the direction given by $(1, 1, 0)$.

(b) Compute the curl and the divergence of the vector field $F + \nabla f$.

length $= \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$

a) $$\text{grad } f = \left[2xz,\ 0,\ x^2 \right]$$

No evaluation here.

Normalize direction: $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ← unit vector

$$D_{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} f = \text{grad } f \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \left[2xz,\ 0,\ x^2 \right] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} 2xz = \boxed{\sqrt{2}\, xz}$$

b) $$F + \nabla f = \left[0,\ e^{xyz},\ 0 \right] + \left[2xz,\ 0,\ x^2 \right]$$

$$= \left[2xz,\ e^{xyz},\ x^2 \right]$$

$$\text{div}(F + \nabla f) = 2z + e^{xyz} xz + 0 = \boxed{z\left(2 + xe^{xyz}\right)}$$

curl?

$$\begin{array}{ccc} \frac{\partial}{\partial x} & 2xz & z \\ \frac{\partial}{\partial y} & e^{xyz} & x \\ \frac{\partial}{\partial z} & x^2 & y \\ \frac{\partial}{\partial x} & 2xz & \end{array}$$

$$\text{Curl}(F + \nabla f) = \left[0 - xye^{xyz},\ 2x - 2x,\ yze^{xyz} - 0 \right]$$

$$= \left[-xye^{xyz},\ 0,\ yze^{xyz} \right]$$

4. A six inch pizza fresh out of the oven has the temperature distribution $98 - 3x^2 - 2y^2 - 3x$ ← T degrees Celsius (the pizza is centered at the origin). Where is the pizza the hottest? Where should you bite first to minimize the chance of burning your mouth?

$T_x = -6x - 3 \quad T_y = -4y \qquad \therefore$ Critical pt @ $\left[-\frac{1}{2}, 0\right]$ $(T = 98.75)$

Boundary : $\underbrace{x^2 + y^2}_{g} = 3^2$ $\qquad$ Lagrange : grad $T = \lambda$ grad g

$$[-6x-3, -4y] = \lambda\, [2x, 2y] \qquad \begin{aligned} -6x - 3 &= 2\lambda x \Rightarrow x = \frac{-3}{2\lambda + 6} \\ -4y &= 2\lambda y \Rightarrow y = 0 \text{ or } \lambda = -2 \end{aligned}$$

If $y = 0$, $x = 3, -3$

If $\lambda = -2$, $x = -\frac{3}{2}$, so $y = \pm\sqrt{3^2 - \frac{3^2}{2^2}} = \pm 3\sqrt{1 - \frac{1}{4}} = \pm \frac{3\sqrt{3}}{2}$

$\therefore$ Critical boundary pts are $[3, 0]$, $[-3, 0]$, $\left[-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right]$, $\left[-\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right]$

with corresponding T values 62, 80, 82.25, 82.25

$\therefore$ Hottest in the middle @ $\left[-\frac{1}{2}, 0\right]$. Bite from the east!

5. Suppose $z = f(u, v)$, where $u = 2x - y$ and $v = x + 2y$. Express the partial derivatives of z with respect to x and y in terms of the partial derivatives of f with respect to u and v.

$$\mathbb{R} \xleftarrow{f} \mathbb{R}^2 \xleftarrow{\begin{bmatrix} u \\ v \end{bmatrix}} \mathbb{R}^2$$

$$z \longleftarrow \begin{bmatrix} u \\ v \end{bmatrix} \longleftarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\underset{[f_x \;\; f_y]}{\overset{\shortparallel}{D(\text{composite})}} = [f_u \;\; f_v] \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \leftarrow D\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)$$

$$= [2f_u + f_v \,,\; -f_u + 2f_v]$$

$$\therefore \quad \begin{aligned} f_x &= 2f_u + f_v \\ f_y &= -f_u + 2f_v \end{aligned}$$