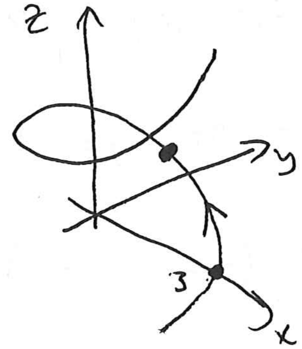
MAT 3243 Midterm 2 Fall 2001

1



$$\gamma = \begin{bmatrix} 3\cos(t) \\ 3\sin(t) \\ 2t \end{bmatrix} \qquad 0 \le t \le \frac{\pi}{2}$$

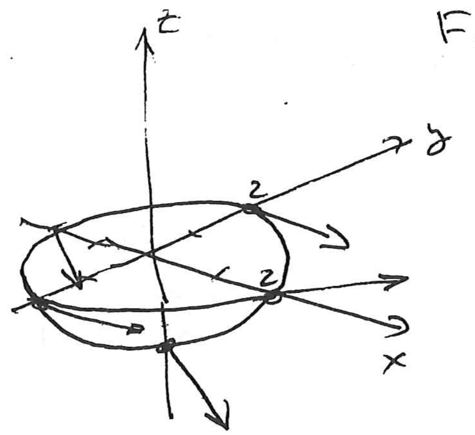
$$d\gamma = \begin{bmatrix} -3\sin(t) \\ 3\cos(t) \\ 2 \end{bmatrix} dt$$

$$|d\gamma| = \sqrt{3^2\sin^2(t) + 3^2\cos^2(t) + 2^2}\ dt$$

$$= \sqrt{3^2+2^2}\ dt = \sqrt{9+4}\ dt = \sqrt{13}\ dt$$

$$\int |d\gamma| = \int_0^{\pi/2} \sqrt{13}\ dt = \boxed{\sqrt{13} \cdot \frac{\pi}{2}}$$

2



$$F(x,y,z) = \begin{bmatrix} 3 \\ x \\ z \end{bmatrix}$$

$$F(2,0,0) = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$F(0,\pm 2,0) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$F(-2,0,0) = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$

$$F(0,0,-2) = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

$$x = 2\sin\varphi\cos\theta$$
$$y = 2\sin\varphi\sin\theta \qquad \frac{\pi}{2} \le \varphi \le \pi$$
$$z = 2\cos\varphi \qquad 0 \le \theta \le 2\pi$$

$$dx = 2\cos\varphi\, d\varphi\cos\theta - 2\sin\varphi\sin\theta\, d\theta$$
$$dy = 2\cos\varphi\, d\varphi\sin\theta + 2\sin\varphi\cos\theta\, d\theta$$
$$dz = -2\sin\varphi\, d\varphi$$

$$d\vec{S} = \begin{bmatrix} dy\,dz \\ dz\,dx \\ dx\,dy \end{bmatrix} = \begin{bmatrix} 4\sin^2\varphi\cos\theta \\ 4\sin^2\varphi\sin\theta \\ 4\cos\varphi\sin\varphi \end{bmatrix} d\varphi\, d\theta = 4\sin\varphi \begin{bmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix} d\varphi\, d\theta$$

$$F\cdot d\vec{S} = 4\sin\varphi\left[3\sin\varphi\underbrace{\cos\theta}_{\text{integrate to 0}} + 2\sin^2\varphi\underbrace{\cos\theta\sin\theta}_{\text{integrate to 0}} + 2\cos^2\varphi\right]d\varphi\, d\theta$$

$$\int_{\pi/2}^{\pi} \sin\varphi\cos^2\varphi\, d\varphi = -\int_0^{-1} u^2\, du = -\frac{u^3}{3}\Big|_0^{-1} = \frac{1}{3}$$

Let $u = \cos\varphi$ $du = -\sin\varphi\, d\varphi$

$$\int F\cdot dS = 8\cdot\frac{1}{3}\cdot 2\pi = \boxed{\frac{16}{3}\pi}$$

(3) By inspection $F = \nabla\Big(\underbrace{\frac{x^2}{2} + x + \frac{y^2}{2} + 2y + \frac{z^2}{2} + 3z}_{\text{potential } u}\Big)$

Therefore by F.T.C. $\displaystyle\int_\gamma F \cdot ds = u \Big|_{\begin{bmatrix}0\\0\\0\end{bmatrix}}^{\begin{bmatrix}1\\-1\\2\end{bmatrix}}$

this depends only on the path's endpoints.

$u\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) = 0$

$u\left(\begin{bmatrix}1\\-1\\2\end{bmatrix}\right) = \frac{1}{2} + 1 + \frac{1}{2} - 2 + 2 + 6 = 8$

$\therefore \quad \displaystyle\int_\gamma F \cdot ds = \boxed{8}$

Alternate Method:

$d\left[(x+1)\,dx + (y+2)\,dy + (z+3)\,dz\right] = 0$

Therefore $\int_\gamma F \cdot ds$ is path independent.

Pick the easiest path: the straight line segment.

Let $\gamma(t) = \begin{bmatrix} t \\ -t \\ 2t \end{bmatrix}$, $0 \le t \le 1$ Then $d\gamma = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} dt$

$$\int_\gamma F \cdot ds = \int_0^1 \begin{bmatrix} t+1 \\ -t+2 \\ 2t+3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} dt = \int_0^1 (t+1+t-2+4t+6)\,dt$$

$$= \int_0^1 (6t+5)\,dt = 3+5 = \boxed{8}$$

4 a) d(2-form on $\mathbb{R}^3$) corresponds to divergence

So $d\omega = \operatorname{div} F \, dx\,dy\,dz = (6z^2 + 6y^2 + 6x^2)\,dx\,dy\,dz$

$= 6\,(x^2+y^2+z^2)\,dx\,dy\,dz$

b) $$\int_{\text{unit sphere}} F \cdot dS = \int_{\substack{\text{unit sphere} \\ \| \\ \partial(\text{unit ball})}} \omega \overset{\text{F.T.C.}}{=} \int_{\text{unit ball}} d\omega$$

$$= \int_{\text{unit ball}} 6(x^2+y^2+z^2)\,dx\,dy\,dz$$

Convert to spherical coords.

$$= 6\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2 \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$\int_0^1 \rho^4\,d\rho = \frac{\rho^5}{5}\Big|_0^1 = \frac{1}{5}$$

$$\int_0^{\pi} \sin\varphi\, d\varphi = -\cos\varphi\Big|_0^{\pi} = 2$$

$$\int_0^{2\pi} d\theta = 2\pi$$

$$\therefore \int_{\text{unit sphere}} F\cdot dS = 6\cdot\frac{1}{5}\cdot 2\cdot 2\pi = \boxed{\frac{24}{5}\pi}$$