① $2n! < n^n$?

$n=1$: $2 < 1$?

$n=2$: $4 < 4$?

$n=3$: $12 < 27$

Conjecture: $\forall n \geq 3 \quad 2n! < n^n$

Pf Induction on n.

Basis $(n=3)$

Let $n > 3$. Assume $\forall k \quad 3 \leq k < n$ we have $2k! < k^k$

In particular, assume $2(n-1)! < (n-1)^{n-1}$
(since $n > 3$, $n-1 \geq 3$)

$$2n! = 2n(n-1)! < n(n-1)^{n-1} < n \cdot n^{n-1} = n^n$$

(since $n-1 < n$)

2.

$$324 = 2 \cdot 148 + 28$$
$$148 = 5 \cdot 28 + 8$$
$$28 = 3 \cdot 8 + 4 = \gcd(324, 148)$$
$$8 = 2 \cdot 4$$

$$4 = 28 - 3 \cdot 8 = 28 - 3(148 - 5 \cdot 28)$$
$$= -3 \cdot 148 + 16 \cdot 28 = -3 \cdot 148 + 16(324 - 2 \cdot 148)$$
$$= 16 \cdot 324 - 35 \cdot 148$$

$$\underbrace{s = 16 \qquad t = -35}_{\text{Bézout}}$$

3. Let $c = \mathrm{lcm}(a,b)$. Then c is a common multiple of a & b, so $\exists\, a', b'$

$$c = aa' = bb'$$

Suppose d is a common multiple of a & b, i.e. $\exists\, a'', b''$

$$d = aa'' = bb''$$

Div. Alg. $\Rightarrow \exists\, q, r$ $\quad d = qc + r$

$$0 \le r < c$$

$$r = d - qc = aa'' - qaa' = a(a'' - qa')$$
$$= bb'' - qbb' = b(b'' - qb')$$

$\therefore$ r is a common multiple of a & b.

Since $r < c$ and $c = \mathrm{lcm}(a,b)$, $r = 0$.

smallest nonzero common multiple.

$\therefore$ $c \mid d$ ☺

4. Suppose $5^{1/3} \in \mathbb{Q}$, i.e.

$\exists\, m, n \in \mathbb{Z}$ $\quad 5^{1/3} = \frac{m}{n}$

Then $n\, 5^{1/3} = m$, so

$$n^3 5 = m^3$$

$\#$ of 5's in $n^3 = 3 \cdot \underbrace{\#\text{ 5's in } n}_{k}$

Similarly for m.

In the equation on the left

$\#$ 5's is $1 \bmod 3$, on the right $0 \bmod 3$ $\ddot\frown$

Alt: we get $3k+1 = 3\ell$

for some k, ℓ.

Then $1 = 3\ell - 3k$, but $3 \nmid 1$ $\ddot\frown$