1. Partition the symmetric group S_3 by left cosets of the cyclic subgroup $\langle(2,3)\rangle$. Do the same with right cosets.

   Let $H{=}\langle(2,3)\rangle$. Since $(2,3)^2 = ()$, $H = \{(), (2,3)\}$.

   Lagrange $\Rightarrow |S_3| = |H|\,[S_3 : H]$. Since $|S_3| = 3! = 6$ and $|H| = 2$, there are $[S_3 : H] = 3$ cosets.

   Left cosets: H; $(1,2)H = \{(1,2),(1,2,3)\}$; $(1,3)H = \{(1,3),(1,3,2)\}$.

   Right cosets: H; $H(1,2) = \{(1,2),(1,3,2)\}$; $H(1,3) = \{(1,3),(1,2,3)\}$.

2. Suppose G, G' are commutative multiplicative groups and $\varphi : G \to G'$ is a surjective homomorphism. For y in G' express its fibre $\varphi^{-1}(y) = \{x \in G : \varphi(x) = y\}$ as a coset of $\ker\varphi$.

   Since φ is onto, there exists $x \in G$ such that $\varphi(x) = y$. Then $\varphi^{-1}(y) = x\ker\varphi$.

   Proof: Let $z \in \ker\varphi$ (i.e. $xz \in x\ker\varphi$). Then $\varphi(xz) = \varphi(x)\varphi(z) = y \cdot 1 = y$, so $xz \in \varphi^{-1}(y)$.

   Conversely, let $x' \in \varphi^{-1}(y)$. Since $x' = xx^{-1}x'$ and $\varphi(x^{-1}x') = \varphi(x)^{-1}\varphi(x') = y^{-1}y = 1$, $x' \in x\ker\varphi$.

3. Find the solution set for the system of congruences

$$35x \equiv 15 \bmod 50$$

$$x \equiv -2 \bmod 30$$

   The first equation is equivalent to $7x \equiv 3 \bmod 10$, which is equivalent to $7x \equiv -7 \bmod 10$, therefore equivalent to $x \equiv -1 \bmod 10$. Thus, any solution must be odd. On the other hand, the second equation implies x must be even. Thus, the solution set is empty.

4. Use Euclid's algorithm for the polynomial ring $\mathbf{R}[x]$ to find the greatest common divisor and the Bézout coefficients for $x^2 + 3x + 2$ and $x^4 + x^3 + 3x + 3$.

   Let $p = x^4 + x^3 + 3x + 3$ and $s = x^2 + 3x + 2$.

   Long division gives $p = qs + r$. where $q = x^2 - 2x + 4$ and $r = -5x - 5$.

   Since $x = -1$ is a root of s, $r|s$. Thus, $\gcd(p, s) = r = -5(x + 1)$.

   NB: Any associate of r (i.e. any nonzero constant multiple of r), such as $x + 1$, is also a valid gcd.

   Solving for r we obtain the Bézout relation is $r = p - qs$.

   $\therefore$ the Bézout coefficients are 1 and $-q = -x^2 + 2x - 4$.

5. Suppose a is a real number and $\varphi\colon \mathbf{R}[x] \to \mathbf{R}$ is the evaluation map $\varphi(p(x)) = p(a)$. Prove that φ is a ring homomorphism. What are its kernel and image?

   Let $p, q \in \mathbf{R}[x]$. Then $\varphi(p + q) = (p + q)(a) = p(a) + q(a) = \varphi(p) + \varphi(q)$ and similarly $\varphi(pq) = (pq)(a) = p(a)q(a) = \varphi(p)\varphi(q)$. Also $\varphi(1) = 1$, so φ is a homomorphism.

   For any $b \in \mathbf{R}$, $\varphi(b) = b$, so the image of f is $\mathbf{R}$.

   For any $q \in \mathbf{R}[x]$, $\varphi((x - a)q(x)) = (a - a)q(a) = 0$, so $(x - a)q(x) \in \ker\varphi$.

   Conversely, if $p \in \ker\varphi$, then $p(a) = 0$, so $x - a$ divides p.

   Therefore, $\ker\varphi = \{(x - a)q(x)\colon q \in \mathbf{R}[x]\} = \langle x - a\rangle$.