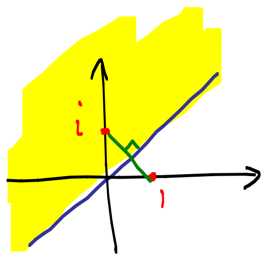
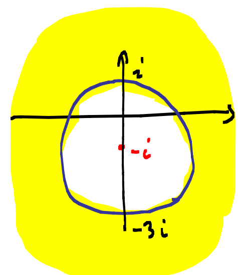
Midterm 1 / 2009.3.5 / MAT 3223.001 / Complex variables

Note Title 3/5/2009

1. Sketch the regions $\{z: \ |z-i| \le |z-1|\}$ and $\{z: \ |z+i| \ge 2\}$.





2. Let $f(z) = |z|^2$. At which z is $f(z)$ complex differentiable? Analytic? Explain.

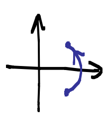
$u = \operatorname{Re} f = |z|^2 = x^2 + y^2 \qquad v = \operatorname{Im} f = 0$

$D\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 0 & 0 \end{bmatrix}$ CR are satisfied $\Leftrightarrow$ $x = y = 0$.

$\therefore$ f is complex differentiable only at the origin.

Since the origin does not contain any neighborhoods, f is analytic nowhere.

3. Integrate $(\operatorname{Re} z + \operatorname{Im} z)\,dz$ along the right half circle centered at 1 from $1-i$ to $1+i$.



$z = 1 + e^{it} \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$dz = ie^{it}\,dt \qquad \operatorname{Re} z = 1 + \cos t = 1 + \frac{e^{it} + e^{-it}}{2}$

$\operatorname{Im} z = \sin t = \frac{e^{it} - e^{-it}}{2i}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + \frac{e^{it}+e^{-it}}{2} + \frac{e^{it}-e^{-it}}{2i}\right) ie^{it}\,dt$$

$\frac{i-1}{2}$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(ie^{it} + \frac{i}{2}e^{2it} + \frac{i}{2} + \frac{1}{2}e^{2it} - \frac{1}{2} \right) dt$$

$\frac{i+1}{2}e^{2it}$

$$= \left[e^{it} + \frac{i+1}{4i}e^{2it} + \frac{i-1}{2}t \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= i - (-i) + \frac{i+1}{4i}\left(\cancel{-1} - (\cancel{-1})\right) + \frac{i-1}{2}\pi = \boxed{2i + \frac{i-1}{2}\pi}$$

Check:

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> z:=1+exp(I*t); (Re(z)+Im(z))*diff(z,t); int(%,t=-Pi/2..Pi/2);
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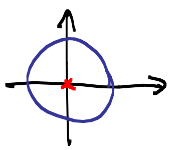
$$z := 1 + e^{(t\,I)}$$

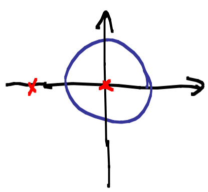
$$(1 + \Re(e^{(t\,I)}) + \Im(e^{(t\,I)}))\,e^{(t\,I)}\,I$$

$$2\,I + \frac{1}{2}\,I\,\pi - \frac{\pi}{2}$$

4. Integrate $\dfrac{\cos(z)}{z^3}\,dz$ and $\dfrac{\cos(z)}{z^2+2z}\,dz$ counterclockwise around the unit circle.



$$\frac{2\pi i}{2!}\cdot \cos(z)''\Big|_{z=0} = \pi i\,(-\cos(0)) = \boxed{-\pi i}$$



$$z^2 + 2z = z(z+2)$$

$$\int \frac{\frac{\cos(z)}{z+2}}{z}\,dz = 2\pi i\,\frac{\cos(0)}{0+2} = \boxed{\pi i}$$

5. Expand $1/z$ in a Taylor series at $z = 1 + i$. What is the disc of convergence?

$$\frac{1}{z} = \frac{1}{1+i+z-(1+i)} = \frac{1}{1+i}\,\frac{1}{1+\frac{z-(1+i)}{1+i}} = \frac{1}{1+i}\sum_{n=0}^{\infty}(-1)^n\left[\frac{z-(1+i)}{1+i}\right]^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(1+i)^{n+1}}\,[z-(1+i)]^n$$

Conv.: $\left|\dfrac{z-(1+i)}{1+i}\right| < 1$

$$|z-(1+i)| < \sqrt{2}$$