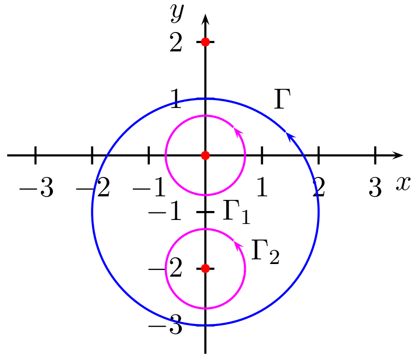
1.



Since $z^2+4=(z-2i)(z+2i)$, the singularities of the integrand are 0 and $\pm 2i$, of which 0 and $-2i$ are inside Γ (see picture).

By the deformation principle (a direct consequence of Cauchy's Theorem), integration along Γ is equivalent to integration along Γ_1 and Γ_2 (see picture), where we can apply Cauchy's Integral Formula.

$$\int_\Gamma \frac{\cos z}{z(z^2+4)}\,dz = \int_{\Gamma_1} \frac{\frac{\cos z}{z^2+4}}{z}\,dz + \int_{\Gamma_2} \frac{\frac{\cos z}{z(z-2i)}}{z+2i}\,dz = 2\pi i\left[\frac{\cos z}{z^2+4}\right]_{z=0} + 2\pi i\left[\frac{\cos z}{z(z-2i)}\right]_{z=-2i}$$

$$= 2\pi i\left[\frac{1}{4}+\frac{\cos(-2i)}{-2i(-2i-2i)}\right] = \pi i\left[\frac{1}{2}-\frac{\cos(-2i)}{4}\right] = \pi i\left[\frac{1}{2}-\frac{\cosh 2}{4}\right]$$

Note that $\cos(-2i)=\frac{1}{2}\left[e^{i(-2i)}+e^{-i(-2i)}\right]=\frac{1}{2}\left[e^2+e^{-2}\right]=\cosh 2$.

2. The origin is the only singularity of the integrand and is inside Γ, so by Cauchy's Integral Formula

$$\int_\Gamma \frac{e^{z^2}}{z^3}\,dz = \frac{2\pi i}{2!}\frac{d^2}{dz^2}\left[e^{z^2}\right]_{z=0} = \pi i\frac{d}{dz}\left[e^{z^2}2z\right]_{z=0} = \pi i\left[e^{z^2}4z^2+e^{z^2}2\right]_{z=0} = 2\pi i$$

3. Parametrize the segment $z=1(1-t)+it=1+(i-1)t$, $0\le t\le 1$. Then $\overline{z}=1+(-i-1)t=1-(i+1)t$ and $dz=(i-1)\,dt$, so

$$\int \overline{z}\,dz = \int_0^1 [1-(i+1)t](i-1)\,dt = \int_0^1 [(i-1)+2t]\,dt = [(i-1)t+t^2]_0^1 = i-1+1=i$$

4. Nonconstant entire functions have dense images, so ...

Claim: f is constant.

Proof: Since $\Re[f(z)]>0$, the function misses the left half-plane and, in particular, misses the open disc of radius 1 centered at -1. In other words, $|f(z)+1|\ge 1$, so $\frac{1}{|f(z)+1|}\le 1$, so $\frac{1}{f(z)+1}$ is a bounded entire function. By Liouville's Theorem it is constant $\frac{1}{f(z)+1}=c$, so $f(z)=\frac{1}{c}-1$ is also constant.